Introduction to Borg-Levinson Inverse Spectral Theory

Éric Soccorsi

Aix-Marseille Université
eric.soccorsi@univ-amu.fr

The study of inverse spectral problems goes back to V.A. Ambarzumian [1] who investigated in 1929 the problem of determining the real potential V appearing in the Sturm–Liouville operator $A = -\partial_{xx} + V$, acting in $L^2(0, 2\pi)$, from partial spectral data of A. He proved in [1] that $V = 0$ if and only if the spectrum of the periodic realization of A equals $\{k^2\ ; k \in \mathbb{N}\}$. For the same operator acting on $L^2(0, \pi)$, but endowed with homogeneous Dirichlet boundary conditions, G. Borg [2] and N. Levinson [8] established that while the Dirichlet spectrum $\{\lambda_k\ ; k \in \mathbb{N}^*\}$ does not uniquely determine V, nevertheless assuming that $\varphi_k'(0) = 1$ for $k \geq 1$, additional spectral data, namely $\{\|\varphi_k\|_{L^2(0,\pi)}\ ; k \in \mathbb{N}^*\}$ is needed, where $\{\varphi_k\ ; k \in \mathbb{N}^*\}$ is an $L^2(0, \pi)$-orthogonal basis of eigenfunctions of A. I.M. Gel'fand and B.M. Levitan [6] proved that uniqueness is still valid upon substituting $\varphi_k'(\pi)$ for $\|\varphi_k\|_{L^2(0,\pi)}$ in the one-dimensional Borg and Levinson theorem.

In 1998, the case where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, was treated by A. Nachman, J. Sylvester and G. Uhlmann [9], and by N.G. Novikov [10]. Inspired by [6], these authors proved that the boundary spectral data $\{(\lambda_k, \partial_\nu \varphi_k);\ k \in \mathbb{N}^*\}$, where $\partial_\nu = \frac{\partial}{\partial_\nu}$ denotes the outward normal derivative on $\partial\Omega$, and (λ_k, φ_k) is the k^{th} eigenpair of A, uniquely determines the Dirichlet realization of the operator A. This result has been improved in several ways by various authors. H. Isozaki [7] extended the result of [9] when finitely many eigenpairs remain unknown. Recently, M. Choulli and P. Stefanov [5] proved stable determination of V from the asymptotic behaviour of $(\lambda_k, \partial_\nu \varphi_k)$ as $k \to \infty$. This result was improved and extended to infinitely extended domains in [4].

This course of three lectures is an introduction to the mathematical analysis of inverse spectral problems of Borg-Levinson type. More precisely, they are concerned with the uniqueness and the stability issues, in the inverse problem of determining the electric potential of the multidimensional Laplace operator by boundary spectral data.

References

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