Global behavior of the Solutions to a Class of Nonlinear, Singular Second Order ODE.

Mama Abdelli

University of Mascara (Algeria)

Laboratoire Analyse et contrôle EDP
Collaboration with Alain Haraux

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Introduction

In this paper we consider the scalar second order ODE

$$\left(|u'|^{l}u'\right)' + c|u'|^{\alpha}u' + d|u|^{\beta}u = 0, \tag{1}$$

where α, β, c, d are positive constants and $l \geq 0$.

This equation was studied in [1] and [2] with nonlinear dissipation $\sigma(t)g(u')$ where σ is a positive fonction, Neumann and Dirichlet boundary conditions, they proved the decay property of the energy without proving a global existence.
We consider a degenerate Kirchhoff equation wave equation with a weak frictional damping,

(P)
$$\begin{cases} (|u_t|^{l-2}u_t)_t - \left(\int_\Omega |\nabla u|^2\, dx \right)^\gamma \Delta u + \alpha(t) g(u_t) = 0 \text{ in } \Omega \times (0, +\infty), \\ u = 0 \text{ on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ on } \Omega, \end{cases}$$

where $l \geq 2$, $\gamma \geq 0$, Ω is a bounded domain in $\mathbb{R}^n$

(H2) Consider $g : \mathbb{R} \to \mathbb{R}$ a nondecreasing C^0 function such that

$$g(v)v > 0 \text{ for all } v \neq 0.$$

and suppose that there exist $c_i > 0\,;\, i = 1,2,3,4$ such that

$$c_1|v|^m \leq |g(v)| \leq c_2|v|^{\frac{1}{m}} \text{ if } |v| \leq 1, \tag{2}$$

$$c_3|v|^s \leq |g(v)| \leq c_4|v|^r \text{ for all } |v| > 1, \tag{3}$$

where $m \geq 1$, $l - 1 \leq s \leq r \leq \frac{n+2}{n-2}$.

In the special case $l = 0$ and $d = 1$ we recover

$$u'' + c|u'|^{\alpha}u' + |u|^{\beta}u = 0. \tag{4}$$

The solutions of (4) are global for $t \geq 0$ and both u and u' decay to 0 as $t \to \infty$. This equation was studied by A. Haraux in [5] by the second author who used some modified energy function to estimate the rate of decay. In addition, he showed that if $\alpha > \frac{\beta}{\beta+2}$ all non-trivial solutions are oscillatory and if $\alpha < \frac{\beta}{\beta+2}$ they are non-oscillatory. In this paper we use some techniques from A. Haraux [5]

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Remark

If $u \not\equiv 0$, at any point t_0 where $u(t_0) \neq 0$ and $u'(t_0) = 0$, the second derivative $u''(t_0)$ does not exist. At least when $\alpha > l - 1$. Indeed for $t \neq t_0$ and $|t - t_0| < \varepsilon$ we have

$$u''(t) = -\frac{c}{l+1}|u'|^{\alpha-l}u'(t) - \frac{d|u|^{\beta}u}{(l+1)|u'(t)|^{l}},$$

hence, as $t \to t_0$, $u''(t)$ has a constant sign and

$$|u''(t)| \to +\infty \quad \text{as} \quad t \to t_0.$$

This implies that u' is not differentiable at t_0.

Existence of solutions

Proposition

Let $(u_0, u_1) \in \mathbb{R}^2$*. The problem (1) has a global solution satisfying*

$$u \in \mathcal{C}^1(\mathbb{R}^+), \quad |u'|^l u' \in \mathcal{C}^1(\mathbb{R}^+) \quad \textit{and} \quad u_0 = u(0), \quad u_1 = u'(0).$$

Proof

To show the existence of the solution for (1), we consider

$$\begin{cases} (\varepsilon + (l+1)|u_\varepsilon'|^l)u_\varepsilon'' + c|u_\varepsilon'|^\alpha u_\varepsilon' + d|u_\varepsilon|^\beta u_\varepsilon = 0 \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1. \end{cases} \tag{5}$$

Here, $\varepsilon > 0$ is a small parameter, devoted to tend to zero.

(1) A priori estimates :
we have the following energy identity :

$$\frac{d}{dt}\left[\frac{\varepsilon}{2}|u_\varepsilon'(t)|^2+\frac{l+1}{l+2}|u_\varepsilon'(t)|^{l+2}+\frac{d}{\beta+2}|u_\varepsilon(t)|^{\beta+2}\right]+c|u_\varepsilon'(t)|^{\alpha+2}=0.$$

By integrating over $(0,t)$, we get

$$\frac{\varepsilon}{2}|u_\varepsilon'(t)|^2+\frac{l+1}{l+2}|u_\varepsilon'(t)|^{l+2}+\frac{d}{\beta+2}|u_\varepsilon(t)|^{\beta+2}+c\int_0^t|u_\varepsilon'(s)|^{\alpha+2}\,ds$$
$$=\frac{\varepsilon}{2}|u_1|^2+\frac{l+1}{l+2}|u_1|^{l+2}+\frac{d}{\beta+2}|u_0|^{\beta+2}.$$

Hence, for some constants M_1, M_2 independent of ε we have

$$\forall t\in[0,T_{\max}),\quad |u_\varepsilon(t)|\le M_1,\;\; |u_\varepsilon'(t)|\le M_2. \tag{6}$$

Now we have

$$\left|\left(|u_\varepsilon'(t)|^l u_\varepsilon'(t)\right)'\right| = (l+1)|u'(t)|^l |u_\varepsilon''(t)| \leq \left|(\varepsilon + (l+1)|u_\varepsilon'(t)|^l) u_\varepsilon''(t)\right|,$$

and by using (5) and (6), we deduce

$$\left|\left(|u_\varepsilon'(t)|^l u_\varepsilon'(t)\right)'\right| \leq M_4. \tag{7}$$

(2) Passage to the limit :
As a consequence of Ascoli's theorem and a priori estimate (6), we may extract a subsequence which is still denoted for simplicity by (u_ε) such that for every $T > 0$

$$u_\varepsilon \to u \quad \text{in} \ \mathcal{C}^1(0,T)$$

as ε tends to 0.

Integrating (5) over $(0,t)$, we then have, as ε tends to 0

$$|u'_\varepsilon|^l u'_\varepsilon \to -c\int_0^t |u'(s)|^\alpha u'(s)\,ds - d\int_0^t |u(s)|^\beta u(s)\,ds + |u'(0)|^l u'(0) \quad \text{in } \mathcal{C}^0(0,T).$$

Hence

$$|u'|^l u' = -c\int_0^t |u'(s)|^\alpha u'(s)\,ds - d\int_0^t |u(s)|^\beta u(s)\,ds + |u'(0)|^l u'(0), \quad (8)$$

and $|u'|^l u' \in \mathcal{C}^1(0,T)$. finally by differentiating (8) we conclude that u is a solution of (1). This completes the proof of Proposition 2.1

Uniqueness of solutions

Proposition

Assume that $l \leq \inf(\alpha, \beta)$. *Then for any interval* J *and any* $\tau \in J$ *, if a solution* u *of (1) satisfies*

$$u \in \mathcal{C}^1(J), \quad |u'|^l u' \in \mathcal{C}^1(J) \quad \text{and} \quad u(\tau) = u'(\tau) = 0,$$

then $u \equiv 0$.

Proposition

Let $a \neq 0$. Then for J an interval containing 0 and such that $|J|$ is small enough, equation (1) has at most one solution satisfying

$$u \in \mathcal{C}^1(J), \quad |u'|^l u' \in \mathcal{C}^1(J) \quad \text{and} \quad u_0 = a, \quad u_1 = 0.$$

Theorem 1

Assume that $l \leq \inf(\alpha, \beta)$. then for any $(u_0, u_1) \in \mathbb{R}^2$, the equation (1) has a unique global solution satisfying

$$u \in \mathcal{C}^1(\mathbb{R}^+), \quad |u'|^l u' \in \mathcal{C}^1(\mathbb{R}^+) \quad \text{and} \quad u_0 = u(0), \quad u_1 = u'(0).$$

The finite number of zero u and u'

Corollary 1

Let $u \in \mathcal{C}^1(J)$ be any solution of (1) with $|u'|^l u' \in \mathcal{C}^1(J), \ u \not\equiv 0$. Then for each compact interval $K \subset J$ the set $F = \{t \in K, \ u(t) = 0\}$ is finite.

Corollary 2

Let $u \in \mathcal{C}^1(J)$ be any solution of (1) with $|u'|^l u' \in \mathcal{C}^1(J), \ u \not\equiv 0$. Then for each compact interval $K \subset J$ the set $G = \{t \in K, \ u'(t) = 0\}$ is finite.

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Theorem 2

Assume that

$$\alpha > \frac{\beta(l+1)+l}{\beta+2} \quad , \qquad \text{or} \tag{9}$$

$$\alpha = \frac{\beta(l+1)+l}{\beta+2} \quad \text{and} \quad c < c_0 = (\beta+2)\left(\frac{(\beta+2)(l+1)}{d(\beta+1)(l+2)}\right)^{\frac{\beta+1}{\beta+2}}. \tag{10}$$

Then, any solution $u(t)$ of (1) which is not identically 0 changes sign on each interval (T,∞) and the same thing is true for $u'(t)$.

Theorem 3

Assume

$$\alpha < \frac{\beta(l+1)+l}{\beta+2}$$

Then any solution $u(t)$ of (1) which is not identically 0 has a finite number of zeros on $(0,\infty)$. Moreover, for t large, $u'(t)$ has the opposite sign to that of $u(t)$ and $u''(t)$ has the same sign as $u(t)$.

Theorem 4

Assume that

$$\alpha = \frac{\beta(l+1)+l}{\beta+2}; \quad c \geq c_0 = (\beta+2)\left(\frac{(\beta+2)(l+1)}{d(\beta+1)(l+2)}\right)^{\frac{\beta+1}{\beta+2}}. \tag{11}$$

Then any solution $u(t)$ of (1) which is not identically 0 has at most one zero on $(0, \infty)$.

Proof

We define a polar coordinate system by

$$\left(\frac{d(l+2)}{(\beta+2)(l+1)}\right)^{\frac{1}{2}}|u|^{\frac{\beta}{2}}u = r(t)\cos\theta(t), \qquad |u'|^{\frac{l}{2}}u' = r(t)\sin\theta(t), \tag{12}$$

where r and θ are two C^1 functions and $r(t) = \left(\frac{l+2}{l+1}E(t)\right)^{\frac{1}{2}} > 0$, where

$$E(t) = \frac{l+1}{l+2}|u'|^{l+2} + \frac{d}{\beta+2}|u|^{\beta+2}. \tag{13}$$

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We define the energy associated to the solution of the problem by the following formula

$$E(t) = \frac{l+1}{l+2}|u'|^{l+2} + \frac{d}{\beta+2}|u|^{\beta+2}. \tag{14}$$

By multiplying equation (1) by u', we obtain that on any interval where u is $\mathcal{C}^2$, $E(t)$ is $\mathcal{C}^1$ with

$$\frac{d}{dt}E(t) = -c|u'|^{\alpha+2} \leq 0. \tag{15}$$

In particular (15) holds, whenever $u'(t) \neq 0$.
Now let t_0 be such that $u'(t_0) = 0$. As a consequence of Corollary 2 there exists $\varepsilon > 0$ such that $u \in \mathcal{C}^2((t_0, t_0 + \varepsilon] \cup [t_0 - \varepsilon, t_0))$. Integrating (15) over (τ, t),

$$E(t) - E(\tau) = -c \int_{\tau}^{t} |u'(s)|^{\alpha+2}\, ds, \quad t_0 < \tau \leq t \leq t_0 + \varepsilon,$$

By letting $\tau \to t_0$, we obtain

$$E(t) - E(t_0) = -c \int_{t_0}^{t} |u'(s)|^{\alpha+2}\, ds,$$

Theorem 5

Assuming $\alpha > l$, there exists a positive constant η such that if u is any solution of (1) with $E(0) \neq 0$

$$\liminf_{t\to+\infty} t^{\frac{l+2}{\alpha-l}} E(t) \geq \eta. \tag{16}$$

Theorem 5

Moreover,

(i) if $\alpha \geq \frac{\beta(1+l)+l}{\beta+2}$, then there is a constant $C(E(0))$ depending on $E(0)$ such that

$$\forall t \geq 1, \quad E(t) \leq C(E(0)) t^{-\frac{l+2}{\alpha-l}},$$

(ii) if $\alpha < \frac{\beta(1+l)+l}{\beta+2}$, then there is a constant $C(E(0))$ depending on $E(0)$ such that

$$\forall t \geq 1, \quad E(t) \leq C(E(0)) t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}.$$

Proof of the upper estimate

we consider the perturbed energy function

$$E_\varepsilon(t) = E(t) + \varepsilon |u|^\gamma u |u'|^l u', \tag{17}$$

where $l > 0$, $\gamma > 0$ and $\varepsilon > 0$. shall chosen as follows : assuming first $(\gamma + 1)(l + 2) \geq \beta + 2$ which reduces to

$$\gamma \geq \frac{\beta - l}{l + 2}$$

We obtain as a consequence of Young's inequality, the existence of $M > 0$ for which

$$\forall t \geq 0, \;\; (1 - M\varepsilon)E(t) \leq E_\varepsilon(t) \leq (1 + M\varepsilon)E(t).$$

therefore, assuming $\varepsilon \leq \frac{1}{2M}$ we achieve

$$\forall t \geq 0, \;\; \frac{1}{2}E(t) \leq E_\varepsilon(t) \leq 2E(t).$$

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Theorem 5 implies the following

Corollary 3

If $\alpha \geq \frac{\beta(l+1)+l}{\beta+2}$, then there is a constant C depending on $E(0)$ such that

$$\forall t \geq 1, \quad |u(t)| \leq Ct^{-\frac{l+2}{(\alpha-l)(\beta+2)}},$$

$$\forall t \geq 1, \quad |u'(t)| \leq Ct^{-\frac{1}{\alpha-l}}.$$

On the other hand Theorem 5 implies the following optimality result

Proposition

Assume either (9) or (10). Then, the results of Corollary 3 are optimal. More precisely, any solution $u \not\equiv 0$ of (1) satisfies

$$\limsup_{t \to +\infty} t^{\frac{l+2}{(\beta+2)(\alpha-l)}} u(t) > 0 \tag{18}$$

$$\limsup_{t \to +\infty} t^{\frac{1}{\alpha-l}} u'(t) > 0. \tag{19}$$

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Asymptotics in the non-oscillatory case

Theorem 6

Assuming $l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$, any solution u of (1) satisfies the following alternative : either there is a positive constant C such that

$$\forall t \geq 1, \quad E(t) \leq C(E(0))t^{-\frac{l+2}{\alpha-l}}, \tag{20}$$

or we have

$$\limsup_{t\to\infty} t^{\frac{(\alpha+1)(\beta+2)}{\beta-\alpha)}} E(t) > 0. \tag{21}$$

In the first case, the solution is called fast, and in the second case slow.

Remark

Assume $l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$. Then if a solution u of (1) satisfies (20) there exists a positive constant C such that

$$\forall t \geq 1, \;\; |u(t)| \leq Ct^{-\frac{l+1-\alpha}{\alpha-l}},$$

and

$$\forall t \geq 1, \;\; |u'(t)| \leq Ct^{-\frac{1}{\alpha-l}}.$$

Theorem 7

Let $l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$. Then, there exists a solution $u > 0$ of (1) such that for some constant $C > 0$

$$\forall t \geq 0, \;\; u(t) \leq C(1+t)^{-\frac{1-\alpha+l}{\alpha-l}}, \quad |u'(t)| \leq C(1+t)^{-\frac{1}{\alpha-l}}.$$

Theorem 8

Let $\alpha < \frac{\beta(l+1)+l}{\beta+2},\ c > 0,\ d > 0$. Then (1) has an open set of initial data leading to a slow solution, which means a solution satisfying (21).

Proof

For any solution u of the equation we introduce the new coordinates (z, w) defined by

$$z = \sqrt{\frac{d(l+2)}{(\beta+2)(l+1)}}|u|^{\frac{\beta}{2}}u, \quad w = |u'|^{\frac{l}{2}}u',$$

and since

$$z' = a|z|^{\frac{\beta}{\beta+2}} w^{\frac{2}{l+2}}, \tag{22}$$

with $a = \frac{d(l+2)}{2(l+1)} \left(\frac{(\beta+2)(l+1)}{d(l+2)} \right)^{\frac{\beta+1}{\beta+2}} > 0$ and

$$w' = -a|w|^{-\frac{l}{l+2}} |z|^{\frac{\beta}{\beta+2}} z - c\frac{l+2}{2(l+1)} |w|^{\frac{2\alpha-l+2}{l+2}} \operatorname{sgn}(w) \tag{23}$$

valid whenever $w \neq 0$. For $u < 0,\ u' > 0$, we consider the region $S_{\varepsilon,M}$

$$S_{\varepsilon,M} = \left\{ (z,w) \in \mathbb{R}^2 / z < 0,\ z^2 + w^2 < \varepsilon^2,\ 0 < \frac{w}{|z|} < M \right\}.$$

For any finite M given in advance, we shall show that for ε small enough, the region $S_{\varepsilon,M}$ is positively invariant. To this end we introduce the vector

$$F(z,w) := \left(a|z|^{\frac{\beta}{\beta+2}} w^{\frac{2}{l+2}}, -a|w|^{-\frac{l}{l+2}} |z|^{\frac{\beta}{\beta+2}} z - c\frac{l+2}{2(l+1)} |w|^{\frac{2\alpha-l+2}{l+2}} \operatorname{sgn}(w) \right)$$

so that as long (z,w) remains in $S_{\varepsilon,M}$ we have the equation

$$(z', w') = F(z,w)$$

Setting $B_\varepsilon = \{(z,w) \in \mathbb{R}^2 / z^2 + w^2 \leq \varepsilon^2\}$, since

$$\langle F(z,w),(z,w)\rangle = -c\frac{l+2}{2(l+1)}|w|^{\frac{2(\alpha+2)}{l+2}} \leq 0,$$

we find that the solution cannot escape $S_{\varepsilon,M}$ at a point of ∂B_ε. By backward uniqueness it is clear that (z,w) cannot leave $S_{\varepsilon,M}$ through $(0,0)$. We now show that if ε is small enough, the solution cannot escape at any point of

$$\triangle_M = \{(-\lambda, M\lambda),\ \lambda \in (0,+\infty)\}$$

lying in the closure of B_ε.

Indeed we have

$$F(-\lambda, M\lambda) = \left(aM^{\frac{2}{l+2}}\lambda^{\frac{\beta}{\beta+2}+\frac{2}{l+2}}, aM^{\frac{-l}{l+2}}\lambda^{\frac{\beta}{\beta+2}+\frac{2}{l+2}} - cM^{\frac{2\alpha-l+2}{l+2}}\frac{l+2}{2(l+1)}\lambda^{\frac{2\alpha-l}{l+2}+\frac{2}{l+2}}\right)$$

Since $\frac{2\alpha-l}{l+2} < \frac{\beta}{\beta+2}$ as a consequence of $\alpha < \frac{\beta(l+1)+l}{\beta+2}$, for λ small enough the field at any point of $\triangle_M$ points into the region $S_{\varepsilon,M}$. And smallness of λ is a consequence of smallness of ε whenever $(z,w) \in \triangle_M$.

Finally, since $F(-\lambda, w)$ tends to $(0, +\infty)$ as $w \to 0$, the solution cannot escape $S_{\varepsilon,M}$ at a point lying on the horizontal axis.

More precisely, assuming the contrary means that for some finite $t_0 > 0$ we have $w > 0$ on $[t_0 - \delta, t_0)$, $w(t_0) = 0,$ and $z(t_0) < 0$. Then for t sufficiently close to t_0 :

$$\begin{aligned} w' &\geq C_1 w^{-\frac{1}{l+2}} - C_2 w^{\frac{2\alpha_l+2}{l+2}} \\ &\geq C w^{-\frac{1}{l+2}} \end{aligned}$$

so that $w(t)$ is increasing for t sufficiently close to t_0, and this contradicts $w(t_0) = 0$.

Finally, for any trajectory of (22) and (23) lying in any region $S_{\varepsilon,M}$, $\frac{w}{|z|} = |\tan\theta|$ is bounded , when for a fast solution $|\tan\theta|$ blows-up at infinity in t. Hence all solutions confined in $S_{\varepsilon,M}$ are slow solutions.

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Thank you