# A general decay result of a wave equation with a dynamic boundary control of diffusive type

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2. Global existence
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# Considered system

In this talk, we consider an initial boundary value problem for the linear wave equation reading as

$$(P)\quad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty) \\ y(0,t) = 0 & \text{in } (0,+\infty) \\ my_{tt}(L,t) + y_x(L,t) = -\zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi & \text{in } (0,+\infty) \\ \partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - y_t(L,t)\mu(\xi) = 0 & \text{in } (-\infty,\infty)\times(0,+\infty) \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L) \\ \phi(\xi,0) = \phi_0 & \text{in } (-\infty,\infty) \end{cases}$$

where $(x,t) \in (0,L)\times(0,+\infty), m > 0, \zeta > 0$ and $\eta \geq 0$. and $\mu$ is a general measure density and the initial data are taken in suitable spaces.
The problem $(P)$ describes the motion of a pinched vibration cable with tip mass $m > 0$.

# Considered system

When $\phi_0 = 0, \mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ and $\zeta = \gamma\pi^{-1}\sin(\alpha\pi)$ with $\gamma > 0$ and $0 < \alpha < 1$, solving $\{(P)_3, (P)_4\}$, we obtain

$$(CF) \qquad my_{tt}(L,t) + y_x(L,t) = -\gamma\partial_t^{\alpha,\eta}y(L,t) \quad \text{on } (0,+\infty),$$

where $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order $\alpha$ $(0 < \alpha < 1)$ with respect to the time variable (see **[6]**). It is defined as follows

$$\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds.$$

# Physical interpretations

For simplicity, the wave speed is chosen to be unity and a subscript letter denotes a partial differential with respect to the corresponding variable. In the above formulation, $(P)_1$ is the wave equation for the cable, $(P)_2$ is the boundary condition at the clamped end, $(P)_3$ is the boundary condition at the free end, $m > 0$ is the tip mass, and $\zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi$ is the nonlocal boundary control force applied at the free end, where $\mu$ is a diffusive representation and $\phi$ is a diffusive state. We call the system $(P)_3$ and $(P)_4$ a dynamic diffusive realization associated with the wave equation and can describe an active dynamic boundary viscoelastic damper designed for the purpose of reducing the vibrations.

# Physical interpretations

The problem arises in many physical applications, e.g. large scale flexible space structures and is interesting from a mathematical point of view, since if perfect contact between the cable and the load is assumed, the ordinary differential equation describing the dynamical process at the end to which the load is attached, becomes a dynamical boundary condition for the partial differential equation for the vibrations of the cable. See for example the early work of

# Energy function

$$E(t) = \frac{1}{2}\|y_t\|_2^2 + \frac{1}{2}\|y_x\|_2^2 + \frac{m}{2}|y_t(L,t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 \, d\xi.$$

# Dissipation of (P)

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi,t)|^2 \, d\xi \leq 0.$$

We have $E' \leq 0$, and then the system $(P)$ is dissipative, where the dissipation is guaranteed by the finite memory term.
If $\zeta = 0$ (no memory term in $(P)$), then $E = E(0)$, and therefore $(P)$ is conservative.

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# Well-posedness

Let $U = (y, y_t, \phi, v)^T, v = \varphi_t(L)$. $(P')$ is equivalent to

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (y_0, y_1, \phi_0, v_0), \end{cases} \tag{1}$$

$$\mathcal{A}\begin{pmatrix} y \\ u \\ \phi \\ v \end{pmatrix} = \begin{pmatrix} u \\ y_{xx} \\ -(\xi^2+\eta)\phi + u(L)\mu(\xi) \\ -\frac{1}{m}y_x(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi \end{pmatrix}$$

# Well-posedness

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (y,u,\phi,v)^T \text{ in } \mathcal{H} : y \in H^2(0,L) \cap H_L^1(0,L), u \in H_L^1(0,L), v \in \mathbb{C}, \\ -(\xi^2+\eta)\phi + u(L)\mu(\xi) \in L^2(-\infty,+\infty), u(L) = v, \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\} \quad (2)$$

where, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = H_L^1(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C}.$$

For $U_1 = U_j = (y_1,u_1,\phi_1,v_1)^T \in \mathcal{H}$, $U_2 = (y_2,u_2,\phi_2,v_2)^T \in \mathcal{H}$, we define the following inner product in $\mathcal{H}$

$$\langle U_1, U_2 \rangle_{\mathcal{H}} = \int_0^L (u_1\overline{u}_2 + y_{1,x}\overline{y}_{2,x})\, dx + \zeta \int_{-\infty}^{+\infty} \phi_1\overline{\phi}_2\, d\xi + m v_1 \overline{v}_2.$$

# Well-posedness

The operator $\mathcal{A}$ generates a $C_0$- semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U = (\varphi, u, \phi, v)^T$. Using the fact that

$$E(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{3}$$

we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2 \, d\xi \tag{4}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $(f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we seek $U = (y, u, \phi, v)^T \in D(\mathcal{A})$ solution of the following system of equations

# Well-posedness

$$\begin{cases} \lambda y - u = f_1, \\ \lambda u - y_{xx} = f_2, \\ \lambda \phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ \lambda v + \frac{1}{m} y_x(L) + \frac{\zeta}{m} \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4. \end{cases} \tag{5}$$

Problem (5) is equivalent to the problem

$$a(y, w) = L(w) \tag{6}$$

where the bilinear form $a : H_L^1(0, L) \times H_L^1(0, L) \to \mathbb{R}$ and the linear form $L : H_L^1(0, L) \to \mathbb{R}$ are defined by

# Well-posedness

$$a(y,w) = \int_0^L (\lambda^2 y\overline{w} + y_x\overline{w}_x)\,dx + \lambda(\lambda m + \tilde{\zeta})y(L)\overline{w}(L)$$

$$L(w) = \int_0^L (f_2 + \lambda f_1)\overline{w}\,dx - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,d\xi\;\overline{w}(L) + (\lambda m + \tilde{\zeta})f_1(L)\overline{w}(L) + mf_4\overline{w}(L).$$

where $\tilde{\zeta} = \zeta\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\,d\xi$.

# Well-posedness

It is easy to verify that $a$ is continuous and coercive, and $L$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H_L^1(0, L)$ problem (6) admits a unique solution $\varphi) \in H_L^1(0, L)$. Applying the classical elliptic regularity, it follows that $\varphi \in H^2(0, L)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, using HilleñYosida theorem, we have the following results.

**Theorem (Existence and uniqueness)**

(1) *If $U_0 \in D(\mathcal{A})$, then system (1) has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) *If $U_0 \in \mathcal{H}$, then system (1) has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

# Plan

# Lack of exponential stability

**Theorem**

*The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

**Proof :** We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system $(P)$ from being exponentially stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let $\lambda$ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (y, u, \phi, v)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$
\begin{cases}
\lambda y - u = 0, \\
\lambda u - y_{xx} = 0, \\
\lambda \phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = 0, \\
\lambda v + \dfrac{1}{m} y_x(L) + \dfrac{\zeta}{m} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0
\end{cases} \tag{7}
$$

From $(7)_1 - (7)_2$ for such $\lambda$, we find

$$
\lambda^2 y - y_{xx} = 0. \tag{8}
$$

Since $v = u(L)$, using $(7)_3$ and $(7)_4$, we get

$$
\begin{cases}
y(0) = 0, \\
\left( \lambda + \dfrac{\zeta}{m} \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \lambda + \eta}\, d\xi \right) u(L) + \dfrac{1}{m} y_x(L)
\end{cases} \tag{9}
$$

The solution $\varphi$ is given by

$$\varphi(x) = \sum_{i=1}^{2} c_i e^{t_i x},\ t_1 = \lambda,\ t_2 = -\lambda. \tag{10}$$

Thus the boundary conditions may be written as the following system :

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

where we have set

$$h(r) = \lambda^2 + \frac{\zeta\lambda}{m} \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi + \frac{r}{m}.$$

Hence a non-trivial solution $\varphi$ exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = \det M(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Let

$$\lambda_k^0 = i\left(\frac{k\pi}{L} + \frac{1}{mk\pi}\right)$$

and

$$S(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi.$$

We have the following Lemma.

**Lemma**

*There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*,|k|\geq N} \subset \sigma(\mathcal{A}) \qquad (12)$$

*where*

$$\lambda_k = \lambda_k^0 - \frac{\zeta L}{m^2k^2\pi^2}\Re S(\lambda_k^0) - i\frac{\zeta L}{m^2k^2\pi^2}\Im S(\lambda_k^0) + o(\Re S(\lambda_k^0)) + i\,o(\Im S(\lambda_k^0)),\ k \geq N,$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N.$$

*Moreover for all $|k| \geq N$, the eigenvalues $\lambda_k$ are simple.*

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues.

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# Asymptotic behavior

**Theorem ([11])**

*Let $X$ be a Hilbert space and let $\mathcal{A}$ be the generator of a bounded $C_0$-semigroup $(S(t))_{t\geq 0}$ on $X$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M(|\beta|),$$

*where $M : \mathbb{R}_+ \to (0, \infty)$ is a continuous non-decreasing function of positive increase. Then, for all initial data $U_0 = (y_0, y_1, \phi_0, v_0) \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of $U_0$, such that*

$$\|e^{\mathcal{A}t}U_0\| \leq C\frac{1}{M^{-1}(t)}\|U_0\|_{D(\mathcal{A})}, \quad t \to \infty.$$

**Remark**

*Theorem 3 extends several theorems obtained by Borichev and Tomilov in* **[1]** *and Batty, Chill and Tomilov in* **[3]**.

### Lemma

*$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

### Theorem

*Let*

$$\Lambda(\lambda) = \frac{|\lambda|^2}{(\Re S(i\lambda))}.$$

*Then, for all initial data $U_0 = (y_0, y_1, \phi_0, v_0) \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of $U_0$, such that the semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ satisfies the following decay estimate*

$$\|e^{\mathcal{A}t}U_0\| \leq C\frac{1}{\Lambda^{-1}(t)}\|U_0\|_{D(\mathcal{A})}, \quad t \to \infty, \tag{13}$$

*where $\Lambda^{-1}$ is any asymptotic inverse of $\Lambda$. Moreover, the rate of energy decay is optimal for general initial data in $D(\mathcal{A})$.*

**Proof**

Let $\lambda \in \mathbb{R}$. Let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ be given, and let $X = (y, u, \phi, v)^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})X = F. \tag{14}$$

Equivalently, we have

$$\begin{cases} i\lambda y - u = f_1, \\ i\lambda u - y_{xx} = f_2, \\ i\lambda \phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ i\lambda v + \frac{1}{m} y_x(L) + \frac{\zeta}{m} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4, \end{cases} \tag{15}$$

From $(15)_1$ and $(15)_2$, we have

$$\lambda^2 y + y_{xx} = -(f_2 + i\lambda f_1)$$

with $\varphi(0) = 0$. Suppose that $\lambda \neq 0$. Then

$$y(x) = c_1 \sin \lambda x - \frac{1}{\lambda} \int_0^x (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \lambda(x - \sigma)\, d\sigma, \tag{16}$$

$$y_x(x) = c_1 \lambda \cos \lambda x - \int_0^x (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \lambda(x - \sigma)\, d\sigma. \tag{17}$$

From $(15)_3$ and $(15)_4$, we have

$$\phi(\xi) = \frac{u(L)\mu(\xi) + f_3(\xi)}{i\lambda + \xi^2 + \eta}$$

$$\left(i\lambda + \frac{\zeta}{m} \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi\right) u(L) + \frac{1}{m} y_x(L) + \frac{\zeta}{m} \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi = f_4. \tag{18}$$

Since

$$u(L) = i\lambda\varphi(L) - f_1(L),$$

using (16), (17) and (18), we get

$$\lambda c_1 \left[ iI \sin\lambda L + \frac{1}{m}\cos\lambda L \right] = J + If_1(L) + iI\int_0^L (f_2(\sigma) + i\lambda f_1(\sigma))\sin\lambda(L-\sigma)\,d\sigma + \frac{1}{m}\int_0^L (f_2(\sigma) + i\lambda f_1(\sigma))\cos\lambda(L-\sigma)\,d\sigma \tag{19}$$

where

$$\begin{aligned} I &= i\lambda + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi, \\ J &= f_4 - \frac{\zeta}{m}\int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi. \end{aligned}$$

We set

$$
\begin{aligned}
g(\lambda) &= iI\sin\lambda L + \frac{1}{m}\cos\lambda L \\
&= -\lambda\sin\lambda L + \frac{1}{m}\cos\lambda L + i\frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda+\xi^2+\eta}\,d\xi\sin\lambda L \\
&= -\lambda\sin\lambda L + \frac{1}{m}\cos\lambda L + i\frac{\zeta}{m}\Re\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda+\xi^2+\eta}\,d\xi\sin\lambda L \\
&\quad -\frac{\zeta}{m}\Im\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda+\xi^2+\eta}\,d\xi\sin\lambda L
\end{aligned}
$$

where $\theta \in ]-\pi/2, \pi/2[$ such that

$$
\begin{aligned}
\cos\theta &= \frac{\eta}{\sqrt{\lambda^2+\eta^2}} \\
\sin\theta &= \frac{\lambda}{\sqrt{\lambda^2+\eta^2}}
\end{aligned}
$$

It is clear that

$$g(\lambda) \neq 0 \quad \forall\lambda \in \mathbb{R}.$$

Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

The constant $c_1$ in (19) satisfies

$$\begin{aligned} |\lambda||c_1||\Xi(\lambda)| &\leq |J| + |I||f_1(L)| + |I| \left| \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \lambda(L-\sigma)\, d\sigma \right| \\ &\quad + \frac{1}{m} \left| \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \lambda(L-\sigma)\, d\sigma \right| \\ &\leq o(1) + c \left( \|f_1\|_{H^1(0,L)} + \|f_2\|_{L^2(0,L)} \right), \end{aligned} \tag{20}$$

As $\Xi(\lambda) \neq 0$ for all $\lambda$, then $c_1$ is uniqueley determined by (20). Hence the operator $i\lambda - \mathcal{A}$ is surjective for all $\lambda$. Moreover, Taking account of Lemma 4.1, the operator $i\lambda - \mathcal{A}$ is injective for all $\lambda$. Then $i\mathbb{R} \subset \varrho(\mathcal{A})$. Moreover, we have

$$|\Xi(\lambda)| \geq c \frac{1}{|\lambda|} (\Re S(i\lambda)).$$

Then, we conclude that

$$|c_1| \leq c \frac{|\lambda|}{(\Re S(i\lambda))}.$$

From (17), $(15)_1$ and (16), we deduce that

$$\|y_x\|_{L^2(0,L)}, \|u\|_{L^2(0,L)}, |v| \leq c\frac{|\lambda|^2}{(\Re S(i\lambda))}\left(\|f_1\|_{H^1(0,L)} + \|f_2\|_{L^2(0,L)}\right).$$

Moreover from (4), we get

$$\int_{-\infty}^{+\infty} |\phi(\xi)|^2 \, d\xi \leq C\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2 \, d\xi \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c\frac{|\lambda|^2}{(\Re S(i\lambda))} \text{ as } |\lambda| \to \infty.$$

Moreover it is easy to see that $\Lambda$ has a positive increase. Thus, the condition of Theorem 4 is now verified, furnishing the validity of (13).

**Example** Let $l : [1, \infty) \to (0, \infty)$ be a slowly varying function such that $\mu(\xi) = |\xi|^\alpha l(|\xi|), -1/2 < \alpha < 1/2$, and let

$$g(s) = \int_0^s \frac{\mu^2(\sqrt{\xi})}{\sqrt{\xi}}\, d\xi, \quad s \geq 0.$$

Then by [[5], Theorem 1.5.8] we see that

$$g(s) \sim \frac{1}{\alpha + 1/2} s^{\alpha+1/2}(l(\sqrt{s}))^2, s \to +\infty$$

and hence $\Re S(i\lambda) \sim \Gamma(1/2 - \alpha)\Gamma(1/2 + \alpha)\lambda^{\alpha-1/2}(l(\sqrt{\lambda}))^2$ as $\lambda \to +\infty$ by [[5], Theorem 1.7.4]. Then

$$\|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{5/2-\alpha}(l(\sqrt{\lambda}))^{-2}.$$

Finally, this estimate is consistent with the asymptotic expansion of eigenvalues in Lemma 3.1. Hence the decay is optimal. That is, this rate cannot be improved for general initial data in $D(\mathcal{A})$.
For the particular case $\mu(\xi) = |\xi|^{(2\varpi-1)/2}$, where $0 < \varpi < 1$, we obtain

$$\|e^{\mathcal{A}t}U_0\| \leq \frac{C}{t^{1/(3-\varpi)}}\|U_0\|_{D(\mathcal{A})}.$$

In **[4]**, by multiplied method, we only obtain
$\|e^{\mathcal{A}t}U_0\| \leq C/t^{1/(4-2\varpi)}\|U_0\|_{D(\mathcal{A})}$ which is less better.

## Remark

*Our method,also, works for the following problem :*

$(EB)$

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ y(0,t) = y_x(0,t) = 0 & \text{in } (0,+\infty), \\ y_{xx}(L,t) = 0 & \text{in } (0,+\infty), \\ y_{xxx}(L,t) = \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi & \text{in } (0,+\infty), \\ \partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - y_t(L,t)\mu(\xi) = 0 & \text{in } (-\infty,+\infty)\times(0,+\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L). \end{cases}$$

*We easily obtain the following Theorem.*

## Theorem

*Let*

$$\Lambda(\lambda) = \frac{1}{(\Re S(i\lambda))}.$$

*Assume that* $\Lambda$ *has a positive increase. Then the semigroup* $S_{\mathcal{A}}(t)_{t\geq 0}$ *associated to* $(EB)$ *satisfies the following decay estimate*

$$\|e^{\mathcal{A}t}U_0\| \leq C\frac{1}{\Lambda^{-1}(t)}\|U_0\|_{D(\mathcal{A})}, \quad t\to\infty,$$

A. Borichev, Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.* **347** (2010)-2, 455-478.

H. Brézis, *Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert,* Notas de Matemàtica (50), Universidade Federal do Rio de Janeiro and University of Rochester, North-Holland, Amsterdam, (1973).

C. J. K. Batty, R. Chill & Y. Tomilov, Fine scales of decay of operator semigroups, *J. Eur. Math. Soc.* **18**, (2016)-4, 853-929.

A. Benaissa & H. Benkhedda, Global existence and energy decay of solutions to a wave equation with a dynamic boundary dissipation of fractional derivative type, *Z. Anal. Anwend.* **37** (2018)-3, 315-339.

N. H. Bingham, C. M. Goldie & J. L. Teugels, *Regular variation,* Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, Cambridge, (1987).

J. U. Choi & R. C. Maccamy, Fractional order Volterra equations with applications to elasticity, J. *Math. Anal. Appl.,* **139** (1989), 448-464.

A. Haraux, *Two remarks on dissipative hyperbolic problems,*