Sharp estimates of the one-dimensional boundary control cost for parabolic systems

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GOAL:

The general aim this talk we will present new results on the cost of the boundary controllability of parabolic systems at time $T > 0$. In particular, we will study sharp estimates of the control cost at time T (T small enough) when the eigenvalues of the generator of the C_0 semigroup accumulate and do not satisfy a gap condition.

Contents

1. The one-dimensional heat equation

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Let us fix $T > 0$. We consider the scalar parabolic problem:

(1) $$\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $y_0 \in H^{-1}(0,\pi)$ is the initial datum, $v \in L^2(0,T)$ is a scalar control function and $y = y(x,t)$ is the state.

Well-posedness:

Theorem

Problem (1) *is well posed: for any $y_0 \in H^{-1}(0,\pi)$ and $v \in L^2(0,T)$, there exists a unique*

$$y \in L^2(Q_T) \cap C^0([0,T]; H^{-1}(0,\pi))$$

solution of (1) *with continuous dependence on the data.*

1. The one-dimensional heat equation

$$(1) \qquad \begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Goal

Analyze the null controllability properties at time $T > 0$ of system (1) and the cost of the null controllability of this system at time $T > 0$ when T is small enough.

Null controllability at time $T > 0$:

Theorem

For any $y_0 \in H^{-1}(0,\pi)$ and any $T > 0$, there exists $v \in L^2(0,T)$ such that the solution y of (1) satisfies $\boxed{y(\cdot,T) = 0}$ in $(0,\pi)$.

1. The one-dimensional heat equation

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In particular, the set

$$\mathcal{Z}_T(y_0) := \{v \in L^2(0,T) : y(T) = 0\} \neq \emptyset.$$

We can then define the **control cost** for system (1) at time T as

$$\mathcal{K}(T) = \sup_{\|y_0\|_{H^{-1}(0,\pi)}=1} \left(\inf_{v\in\mathcal{Z}_T(y_0)} \|v\|_{L^2(0,T)} \right), \quad \forall T > 0,$$

Theorem

There exist $\tau_0 > 0$ and $C_0, C_1 > 0$ such that

$$\exp\left(\frac{C_0}{T}\right) \leq \mathcal{K}(T) \leq \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0,\tau_0).$$

1. The one-dimensional heat equation

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We will apply the **moment method**: Let φ be a solution of the **adjoint problem**:

$$\begin{cases} -\varphi_t - \varphi_{xx} = 0 & \text{in } (0,\pi)\times(0,T), \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 \in H_0^1(0,\pi)^2 & \text{in } (0,\pi). \end{cases}$$

If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0\rangle - \langle y_0, \varphi(0)\rangle = \int_0^T v(t)\varphi_x(0,t)\,dt$$

Thus $\boxed{y(T) = 0} \iff \exists \color{red}{v} \in L^2(0,T)$ such that

$$\boxed{\int_0^T \color{red}{v}(t)\varphi_x(0,t)\,dt = -\langle y_0, \varphi(0)\rangle\,, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2}$$

Moment Method (Fattorini-Russell)

- $\sigma(-\partial_{xx}^2) = \{k^2\}_{k\geq 1} := \{\color{red}{\lambda_k}\}_{k\geq 1}$.
- $\{\color{red}{\Phi_k}\}$ a orthogonal basis of $H_0^1(0,\pi)$, where $\color{red}{\Phi_k} = \sqrt{2/\pi}\sin kx$ are eigenfunctions of the operator $\boxed{-\partial_{xx}^2}$.

Thus $\boxed{y(T)=0} \iff \exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(t)\varphi_x(0,t)\,dt = -\langle y_0, \varphi(0)\rangle\,, \quad \forall \varphi_0 \in H_0^1(0,\pi)^2}$$

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Thus $\boxed{y(T)=0} \iff \exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(t)\varphi_{k,x}(0,t)\,dt = -\langle y_0, \varphi_k(0)\rangle\,, \quad \forall k \geq 1 \ (\varphi_0 = \Phi_k)}$$

1. The one-dimensional heat equation

Choosing $\varphi_0 = \Phi_k$, we have $\varphi_k(\cdot, t) = e^{-\lambda_k(T-t)}\Phi_k$ and

$$\varphi_k(x,0) = e^{-\lambda_k T}\Phi_k(x), \quad \varphi_{k,x}(0,t) = \sqrt{\frac{2}{\pi}} k e^{-\lambda_k(T-t)}$$

Thus $\boxed{y(T) = 0} \iff \exists v \in L^2(0,T)$ such that

$$\boxed{\sqrt{\frac{2}{\pi}} k \int_0^T v(T-t) e^{-\lambda_k t}\, dt = -e^{-\lambda_k T} \left\langle y_0, \Phi_k \right\rangle, \quad \forall k \geq 1}$$

$\iff \exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(T-t) e^{-\lambda_k t}\, dt = -\sqrt{\frac{\pi}{2}} \frac{1}{k} e^{-\lambda_k T} \left\langle y_0, \Phi_k \right\rangle \equiv e^{-\lambda_k T} c_k, \quad \forall k \geq 1}$$

1. The one-dimensional heat equation

$$(1) \qquad \boxed{\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

Summarizing: Moment problem

System (1) is null controllable at time $T > 0$ if and only if for any $y_0 \in H^{-1}(0,\pi)$ there exists $v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(T-t)e^{-\lambda_k t}\,dt = e^{-\lambda_k T} c_k, \quad \forall k \geq 1}$$

where $c_k = c_k(y_0)$ is such that $\sum_{k\geq 1} c_k^2 \leq C\|y_0\|^2_{H^{-1}(0,\pi)}$ ($\lambda_k = k^2$).

In our case $\lambda_k = k^2$. This implies

$$\boxed{\sum_{k\geq 1} \frac{1}{\lambda_k} < \infty}$$

and we deduce that the set $\left\{e^{-\lambda_k t}\right\}_{k\geq 1} \subset L^2(0,T)$ is **minimal** (linearly independent) and:

Theorem

*Under the previous assumptions, $\left\{e^{-\lambda_k t}\right\}_{k\geq 1} \subset L^2(0,T)$ admits a **biorthogonal family** $\{q_k\}_{k\geq 1}$ in $L^2(0,T)$, i.e.:*

$$\int_0^T e^{-\lambda_k t} q_l(t)\, dt = \delta_{kl}, \quad \forall k, l \geq 1$$

1. The one-dimensional heat equation

A formal solution to

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = e^{-\lambda_k T}c_k, \quad \forall k \geq 1,$$

is v given by: $v(T-t) = \sum_{k\geq 1} c_k e^{-\lambda_k T} q_k(t)$,

Question: $v \in L^2(0,T)$?, i.e., is the series $\sum_{k\geq 1} c_k e^{-\lambda_k T} q_k(t)$ convergent in $L^2(0,T)$?

But this question itself amounts to:

$$\|q_k\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

1. The one-dimensional heat equation

Theorem (Fattorini-Russell)

There exists $\tau_0 > 0$ and $C > 0$ such that, if $T \in (0, \tau_0)$, one has

$$\|q_k\|_{L^2(0,T)} \leq Ce^{C\sqrt{\lambda_k}}e^{C/T}, \quad \forall k \geq 1.$$

Theorem (Fattorini-Russell)

There exists $\tau_0 > 0$ and $C > 0$ such that, if $T \in (0, \tau_0)$, one has

$$\|q_k\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_k}} e^{C/T}, \quad \forall k \geq 1.$$

Recall, $v(T-t) = \sum_{k\geq 1} c_k e^{-\lambda_k T} q_k(t)$. Then,

$$\|v\|_{L^2(0,T)} \leq C e^{C/T} \sum_{k\geq 1} |c_k| e^{-\lambda_k T} e^{C\sqrt{\lambda_k}}$$

$$\leq C e^{C/T} \|y_0\|_{H^{-1}(0,T)} \left(\sum_{k\geq 1} e^{-2\lambda_k T} e^{2C\sqrt{\lambda_k}} \right)^{1/2}$$

But, $\boxed{2C\sqrt{\lambda_k} \leq C^2/T + T\lambda_k}$. Thus, for a new constant $C > 0$,

$$\|v\|_{L^2(0,T)} \leq C e^{C/T} \|y_0\|_{H^{-1}(0,T)} \left(\sum_{k\geq 1} e^{-\lambda_k T} \right)^{1/2} \leq \frac{C}{\sqrt{T}} e^{C/T} \|y_0\|_{H^{-1}(0,T)}$$

Then, we have proved

Theorem

There exist $\tau_0 > 0$ *and* $C_1 > 0$ *such that*

$$\mathcal{K}(T) \leq \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0, \tau_0).$$

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$$\mathcal{K}(T) \leq \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0, \tau_0).$$

We have used

1. $\{\lambda_k\}_{k\geq 1} \subset (0, \infty)$ is an increasing sequence and
$$\sqrt{\lambda_k} = K(n+\alpha)\left[1 + O(1/n)\right].$$
2. Gap condition: $|\lambda_k - \lambda_n| \geq \rho \left|k^2 - n^2\right|$, for any $k, n \geq 1$.

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Bibliography: [FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971)

2. A boundary controllability problem for a parabolic system

2. Boundary controllability of a parabolic system

Let us consider the 2×2 linear reaction-diffusion system

$$(2) \qquad \boxed{\begin{cases} y_t - y_{xx} + A_1 y = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}}$$

Here $y = (y_1, y_2)^t$, $y_0 \in H^{-1}((0,\pi);\mathbb{R}^2)$, $v \in L^2(0,T)$ and

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Moment Method (Fattorini-Russell)

- $\sigma(-\partial_{xx}^2 + A_1^t) = \{k^2 \pm i\}_{k \geq 1} := \{\lambda_k\}_{k \geq 1}$.
- $\{\Phi_k\}$ a Riesz basis of $H_0^1(0,\pi;\mathbb{C}^2)$, where Φ_k are eigenfunctions of the operator $\boxed{-\partial_{xx}^2 + A_1^t}$.

2. Boundary controllability of a parabolic system

$$(2) \qquad \begin{cases} y_t - y_{xx} + A_1 y = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

One can prove:

Theorem

System (2) *is null controllable at time* $T > 0$*, for any* $T > 0$.

Proof: The proof is very technical and use the moment method. See, [FERNÁNDEZ-CARA, G.-B., DE TERESA], *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010).

2. Boundary controllability of a parabolic system

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Remark

The eigenvalues of the operator $-\partial_{xx}^2 + A_1^t$ are complex but satisfy ($\delta, \rho > 0$):

1. $\displaystyle\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty; \quad \Re(\lambda_k) \geq \delta |\lambda_k|, \quad \forall k \geq 1;$
2. $|\lambda_k - \lambda_n| \geq \rho |k - n|$, for all $k, n \geq 1$ (**gap condition**).

2. Boundary controllability of a parabolic system

$$(2) \qquad \begin{cases} y_t - y_{xx} + A_1 y = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

As before, the set

$$\mathcal{Z}_T(y_0) := \{v \in L^2(0,T) : y(T) = 0\} \neq \emptyset.$$

The **control cost** for system (2) at time T is now given by

$$\mathcal{K}(T) = \sup_{\|y_0\|_{H^{-1}(0,\pi)}=1} \left(\inf_{v \in \mathcal{Z}_T(y_0)} \|v\|_{L^2(0,T)} \right), \quad \forall T > 0.$$

Question:

$$\mathcal{K}(T) \leq \exp\left(\frac{C}{T}\right), \quad \forall T \in (0,\tau_0)?? \quad (\tau_0 > 0)$$

2. Boundary controllability of a parabolic system

$$(2) \quad \begin{cases} y_t - y_{xx} + A_1 y = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Null controllability of (2) at time $T > 0$:

Moment problem

System (2) is null controllable at time $T > 0$ if and only if for any $y_0 \in H^{-1}(0,\pi;\mathbb{C}^2)$ there exists $v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(T-t) e^{-\lambda_k t}\, dt = e^{-\lambda_k T} c_k, \quad \forall k \geq 1}$$

where $c_k = c_k(y_0)$ is such that $\sum_{k\geq 1} c_k^2 \leq C \|y_0\|^2_{H^{-1}(0,\pi;\mathbb{C}^2)}$.

The assumptions

$$\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty; \quad \Re(\lambda_k) \geq \delta|\lambda_k| \text{ and } |\lambda_k - \lambda_n| \geq \rho|k-n|, \quad \forall k, n \geq 1$$

imply the set $\{e^{-\lambda_k t}\}_{k\geq 1} \subset L^2(0,T)$ is **minimal** (linearly independent) and:

Theorem

*Under the previous assumptions, $\{e^{-\lambda_k t}\}_{k\geq 1} \subset L^2(0,T)$ admits a **biorthogonal family** $\{q_k\}_{k\geq 1}$ in $L^2(0,T)$, i.e.:*

$$\int_0^T e^{-\lambda_k t} q_l(t)\, dt = \delta_{kl}, \quad \forall k, l \geq 1,$$

which satisfies: for any $\varepsilon > 0$, there exists $C_\varepsilon = C_\varepsilon(T) > 0$ such that

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon \Re(\lambda_k)}, \quad \forall k \geq 1.$$

Again, a formal solution to

$$\boxed{\int_0^T v(T-t)e^{-\lambda_k t}\,dt = e^{-\lambda_k T}c_k, \quad \forall k \geq 1,}$$

is v given by: $\boxed{v(T-t) = \sum_{k\geq 1} c_k e^{-\lambda_k T} q_k(t)}$. Taking $\varepsilon = T/2$, we deduce

$$\|v\|_{L^2(0,T)} \leq C_T \sum_{k\geq 1} |c_k| e^{-\Re(\lambda_k)T} e^{\Re(\lambda_k)T/2} \leq C_T \|y_0\|_{H^{-1}(0,T)}$$

Question:

Dependence of C_T with respect to T when $T > 0$ is small??
To this end, we will add some assumptions to the sequence $\{\lambda_k\}_{k\geq 1}$.

Assumptions

$\{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ such that

1. $\boxed{\Re(\lambda_k) > 0}$ and $\exists \beta > 0$ s.t. $\boxed{\Im(\lambda_k) \leq \beta\sqrt{\Re(\lambda_k)}}$, $\forall k \geq 1$.
2. **Gap condition**: for some $\rho, q > 0$,

$$\begin{cases} |\lambda_k - \lambda_n| \geq \rho |k^2 - n^2|, & \forall k, n : |k-n| \geq q. \\ \inf_{k\neq n : |k-n|<q} |\lambda_k - \lambda_n| > 0. \end{cases}$$

3. For some $p, \alpha > 0$,

$$\left| p\sqrt{r} - \mathcal{N}(r) \right| \leq \alpha, \quad \forall r > 0,$$

where $\mathcal{N}$ is the counting function associated with the sequence $\{\lambda_k\}_{k\geq 1}$:

$$\mathcal{N}(r) = \#\{k : |\lambda_k| \leq r\}, \quad \forall r > 0.$$

2. Boundary controllability of a parabolic system

Under the previous assumptions, one has:

Theorem

There exist $\tau_0 > 0$ *and* $C > 0$ *such that, if* $T \in (0, \tau_0)$*, one has*

$$\|q_k\|_{L^2(0,T)} \leq C e^{C\sqrt{\Re(\lambda_k)}} e^{C/T}, \quad \forall k \geq 1.$$

[Benabdallah, Boyer, G.-B., Olive], SIAM J. Control Optim. **52** (2014), no. 5, 2970–3001.

Gap condition

Let us see a physical example where the condition: *"for some* $\rho, q > 0$,

$$\begin{cases} |\lambda_k - \lambda_n| \geq \rho |k^2 - n^2|, \quad \forall k, n : |k-n| \geq q. \\ \inf_{k \neq n : |k-n| < q} |\lambda_k - \lambda_n| > 0." \end{cases}$$

does not hold, i.e., a sequence for which

$$\inf_{k \neq n : |k-n| < q} |\lambda_k - \lambda_n| = 0.$$

3. The boundary controllability of a phase-field system

3. The boundary controllability of a phase-field system

Phase-field system: it is a model describing the transition between the solid and liquid phases in solidification/melting processes of a material occupying a domain.

3. The boundary controllability of a phase-field system

Fix $T > 0$. Notation: $Q_T := (0,\pi) \times (0,T)$

$$(3) \quad \begin{cases} \tilde\theta_t - \xi\tilde\theta_{xx} + \dfrac{1}{2}\rho\xi\tilde\phi_{xx} + \dfrac{\rho}{\tau}\tilde\theta = f(\tilde\phi) & \text{in } Q_T, \\ \tilde\phi_t - \xi\tilde\phi_{xx} - \dfrac{2}{\tau}\tilde\theta = -\dfrac{2}{\rho} f(\tilde\phi) & \text{in } Q_T, \\ \tilde\theta(0,\cdot) = v,\ \tilde\phi(0,\cdot) = c,\ \tilde\theta(\pi,\cdot) = 0,\ \tilde\phi(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde\theta(\cdot,0) = \tilde\theta_0,\ \tilde\phi(\cdot,0) = \tilde\phi_0 & \text{in } (0,\pi). \end{cases}$$

$\boxed{\tilde\theta = \tilde\theta(x,t)}$: the temperature of the material;

$\boxed{\tilde\phi = \tilde\phi(x,t)}$: phase-field function used to identify the solidification level of the material; $\boxed{c \in \{-1,0,1\}}$;

$\boxed{f}$: nonlinear term which comes from the derivative of the classical regular double-well potential W: $f(\tilde\phi) = -\dfrac{\rho}{4\tau}\left(\tilde\phi - \tilde\phi^3\right)$.

$\boxed{\rho > 0, \tau > 0, \xi > 0}$: latent heat, relaxation time; thermal diffusivity.

$\boxed{v \in L^2(0,T)}$: control. $\boxed{\tilde\theta_0, \tilde\phi_0}$: initial data.

3. The boundary controllability of a phase-field system

The phase function $\tilde{\phi}$ describes the phase transition of the material (solid or liquid): $\boxed{\tilde{\phi} = 1}$ solid state of the material; $\boxed{\tilde{\phi} = -1}$ liquid state.

G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal. **92** (1986), no. 3, 205–245.

3. The boundary controllability of a phase-field system

Let us consider the corresponding linear system

$$(4)\qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T := (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y = (\theta, \phi)$,

$$D = \begin{pmatrix} \xi & -\frac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\rho}{\tau} & -\frac{\rho}{2\tau} \\ -\frac{2}{\tau} & \frac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proposition

Assume $y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$. Then, system (4) admits a unique solution $y = (\theta,\phi) \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1})$ which depends continuously on the data.

$$(4)\qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Theorem (Approximate controllability)

Fix $T > 0$. Then, system (4) is approximately controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time T if and only if

$$(5)\qquad \boxed{\xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0}, \quad \forall k,\ell \geq 1, \quad \ell > k.$$

3. The boundary controllability of a phase-field system

$$(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Null controllability with a bound of the control cost?? Consider the vectorial operator $\boxed{L = -D\partial_{xx} + A}$

Moment method

We will use:

- the eigenvalues of L and L^*: $\boxed{\lambda_k^{(1)},\ \lambda_k^{(2)}}$:

$$\boxed{\lambda_k^{(i)} \sim \xi k^2 + \frac{\rho+1}{2\tau} + O_i(k) > 0}$$

- the eigenfunctions of L: $\boxed{\Psi_k^{(1)},\ \Psi_k^{(2)}}$; (basis of $H^{-1}(0,\pi;\mathbb{R}^2)$).
- the eigenfunctions of L^*: $\boxed{\Phi_k^{(1)},\ \Phi_k^{(2)}}$; (basis of $H_0^1(0,\pi;\mathbb{R}^2)$).

Two ingredients:

1. $\boxed{\sum_{k\geq 1}\left(\frac{1}{\lambda_k^{(1)}}+\frac{1}{\lambda_k^{(2)}}\right)<\infty} \Longrightarrow \exists$ a family (not unique) $\left\{q_k^{(1)},q_k^{(2)}\right\}_{k\geq 1}\subset L^2(0,T)$ s.t. $\int_0^T e^{-\lambda_k^{(i)}t}q_\ell^{(j)}\,dt=\delta_{k\ell}\delta_{ij}, \forall k,\ell\geq 1,$ $i,j=1,2$ (**biorthogonality**). $\boxed{\lambda_k^{(i)}\sim \xi k^2+\frac{\rho+1}{2\tau}+O_i(k)>0}$.

3. The boundary controllability of a phase-field system

Two ingredients:

1. $$\boxed{\sum_{k\geq 1}\left(\frac{1}{\lambda_k^{(1)}}+\frac{1}{\lambda_k^{(2)}}\right)<\infty}\implies \exists \text{ a family (not unique)}$$
$$\left\{q_k^{(1)},q_k^{(2)}\right\}_{k\geq 1}\subset L^2(0,T) \text{ s.t. } \int_0^T e^{-\lambda_k^{(i)}t}q_\ell^{(j)}\,dt=\delta_{k\ell}\delta_{ij},\ \forall k,\ell\geq 1,$$
$i,j=1,2$ (**biorthogonality**). $\boxed{\lambda_k^{(i)}\sim \xi k^2+\frac{\rho+1}{2\tau}+O_i(k)>0}$.
2. **Separability:** For some $\rho, q>0$,
$$\begin{cases} |\lambda_k-\lambda_n|\geq \rho|k^2-n^2|, & \forall k,n: |k-n|\geq q. \\ \displaystyle\inf_{k\neq n:|k-n|<q}|\lambda_k-\lambda_n|>0. \end{cases}$$

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2. **Separability:** For some $\rho, q > 0$,

$$\begin{cases} |\lambda_k - \lambda_n| \geq \rho|k^2-n^2|, \quad \forall k,n : |k-n| \geq q. \\ \inf_{k\neq n : |k-n|<q} |\lambda_k - \lambda_n| > 0. \end{cases}$$

   Not in general. If for $j \geq 1$ one has $\xi = \frac{1}{j^2}\frac{\rho}{\tau}$, then $\boxed{\lambda_k^{(2)} = \lambda_{k+j}^{(1)} + O(\frac{1}{k})}$ for k large enough.

Remark: Minimal time

If $\boxed{\xi = \frac{1}{j^2}\frac{\rho}{\tau}}$, for some $j \geq 1$, then, the eigenvalues of L concentrate:

$$\boxed{\inf_{k,\ell\geq 1, k\neq \ell} |\lambda_k - \lambda_\ell| = 0}.$$

In this case the controllability problem for system (4) could have a minimal time $T_0 \in [0, \infty]$ of null controllability which is related to the condensation index of the sequence. In our case, we can prove that $\boxed{T_0 = 0}$, but **control cost?**

4. Bounds on biorthogonal families to complex exponentials

4. Bounds on biorthogonal families to complex exponentials

Let us fix $\{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ such that $\lambda_k \neq \lambda_n$ with $k \neq n$ and

Assumptions:

1. $\Re(\lambda_k) > 0$ and $\Im(\lambda_k) \leq \beta\sqrt{\Re(\lambda_k)}$, for any $k \geq 1$ ($\beta \geq 0$).
2. $\exists \rho, q > 0$ such that

$$\boxed{|\lambda_k - \lambda_n| \geq \rho|k^2 - n^2|}, \quad \forall k, n \geq 1 : |k - n| \geq q.$$

3. $\exists p, \alpha > 0$ such that

$$|p\sqrt{r} - \mathcal{N}(r)| \leq \alpha, \quad \forall r > 0$$

where $\mathcal{N}$ is the counting function associated with the sequence $\{\lambda_k\}_{k\geq 1}$:

$$\mathcal{N}(r) = \#\{k : |\lambda_k| \leq r\}, \quad \forall r > 0.$$

The main result:

Theorem

Under the previous assumptions, there exist $\tau_0 > 0$ and $C > 0$ such that for any $T \in (0, \tau_0)$ there exists a biorthogonal family $\{q_k\}_{k\geq 1}$ to $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$ such that

$$\|q_k\|_{L^2(0,T)} \leq C e^{C\sqrt{\Re(\lambda_k)}} e^{C/T} \prod_{1\leq |k-n| \leq q} |\lambda_k - \lambda_n|^{-1}, \quad \forall k \geq 1$$

Work in progress in collaboration with:

Lydia Ouaili (Aix-Marseille Université).

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Remark

Under the previous assumptions, the corresponding controllability problem has, in general, a minimal time of null controllability $T_0 \in [0, \infty]$.

Theoretical result? Examples?

First example: Phase-field system:

$$(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T := (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y = (\theta, \phi)$,

$$D = \begin{pmatrix} \xi & -\frac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\rho}{\tau} & -\frac{\rho}{2\tau} \\ -\frac{2}{\tau} & \frac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In this case, $\boxed{\lambda_k^{(i)} \sim \xi k^2 + \dfrac{\rho+1}{2\tau} + O_i(k) > 0}$, $q = 1$ and $\prod_{1\le|k-n|\le q} |\lambda_k - \lambda_n|^{-1} \le C\sqrt{\lambda_k}$. Then, for a new constant $C > 0$,

$$\boxed{\|q_k\|_{L^2(0,T)} \le C e^{C\sqrt{\Re(\lambda_k)}} e^{C/T}, \quad \forall k \ge 1}.$$

First example: Phase-field system:

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Reasoning as in the case of the heat equation one has:

Theorem

System (4) is null controllable at any time $T > 0$ and there exist $\tau_0 > 0$ and $C_0 > 0$ such that

$$\mathcal{K}(T) \leq \exp\left(\frac{C_0}{T}\right), \quad \forall T \in (0,\tau_0).$$

Second example: Let us consider the problem

$$(6)\qquad \begin{cases} y_t - y_{xx} + a(x)A_0 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $a \in L^2(0,\pi)$ is a function such that $\int_0^\pi a(x)\,dx = 0$, $y = (y_1, y_2)^t$, $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ and

$$\boxed{A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

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$$\boxed{A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

If $L = -\partial_{xx} + a(x)A_0$, with $D(L) = H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)$, one has

$$\sigma(L) = \sigma(L^*) = \{k^2, k^2 + \beta_k\} \quad \text{where} \quad \{\beta_k\}_{k\geq 1} \in \ell^2.$$

Second example:

$$
(6) \qquad \begin{cases} y_t - y_{xx} + a(x) A_0 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

Lemma

Given $\gamma \in (0,1)$, *there exists* $a \in L^2(0,\pi)$ *with* $\int_0^\pi a(x)\,dx = 0$ *such that*

$$
\sigma(L) = \sigma(L^*) = \{k^2, k^2 + e^{-k^{2\gamma}}\} \quad \textit{where} \quad L = -\partial_{xx} + a(x) A_0.
$$

In this case, $q = 1$ and $\prod_{1 \le |k-n| \le q} |\lambda_k - \lambda_n|^{-1} \le C e^{k^{2\gamma}}$. Then, for a new constant $C > 0$,

$$
\boxed{\|q_k\|_{L^2(0,T)} \le C e^{C\sqrt{k^2}} e^{k^{2\gamma}} e^{C/T}, \quad \forall k \ge 1}.
$$

Second example:

$$(6) \qquad \begin{cases} y_t - y_{xx} + a(x)A_0 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

In this case, we can prove,

Theorem

Under the previous conditions, system (6) is null controllable at any time $T > 0$ and there exist $\tau_0 > 0$ and $C_0, C_1 > 0$ such that

$$\exp\left(\frac{C_0}{T} + \frac{C_0}{T^{\frac{\gamma}{1-\gamma}}}\right) \le \mathcal{K}(T) \le \exp\left(\frac{C_1}{T} + \frac{C_1}{T^{\frac{\gamma}{1-\gamma}}}\right), \quad \forall T \in (0,\tau_0).$$

[Ouaili, G.-B.,] to appear.

4. Bounds on biorthogonal families to complex exponentials

Third example: Let us consider the problem

$$(7) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \operatorname{diag}(1,d)$ ($d \neq 1$), $y = (y_1, y_2)^t$, $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ and

$$\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

If $L = -D\partial_{xx} + A_0$, with $D(L) = H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)$, one has

$$\sigma(L) = \sigma(L^*) = \{k^2, dk^2 : k \geq 1\} = \{\lambda_k\}_{k\geq 1}.$$

Necessary condition when $d \neq 1$.

If system (7) is null controllable at time $T > 0$, then $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

Third example:

$$(7)\quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Theorem (Controllability result $d \neq 1$)

1. $\forall T > 0$: ***Approximate controllability*** *at time T if and only if* $\boxed{\sqrt{d} \notin \mathbb{Q}}$.
2. *Assume* $\boxed{\sqrt{d} \notin \mathbb{Q}}$, $\exists T_0 \in [0, +\infty]$ *such that*
   1. *System (7) is null controllable at time T if* $\boxed{T > T_0}$
   2. *If* $\boxed{T < T_0}$ *the system is* ***not null controllable*** *at time T.*

Ammar Khodja, Benabdallah, G.-B., de Teresa, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal. **267** (2014).

Third example:

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In this case, we can prove,

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$$\sigma(L) = \sigma(L^*) = \{k^2, dk^2 : k \geq 1\} = \{\lambda_k\}_{k\geq 1}.$$

In this case, we can prove,

Theorem

Given $\gamma \in (0,1)$, there exists $d > 0$, with $\sqrt{d} \notin \mathbb{Q}$, such that system (7) is null controllable at any time $T > 0$ and there exist $\tau_0 > 0$ and $C_1 > 0$ such that

$$\mathcal{K}(T) \leq \exp\left(\frac{C_1}{T} + \frac{C_1}{T^{\frac{\gamma}{1-\gamma}}}\right), \quad \forall T \in (0,\tau_0).$$

[OUAILI, G.-B.,] to appear.

Thank you for your attention!!