MULTIDIMENSIONAL BORG-LEVINSON INVERSE SPECTRAL THEORY

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Abstract. This text deals with the multidimensional Borg-Levinson theorem, claiming that the potential of the Dirichlet Laplacian acting in a bounded domain of $\mathbb{R}^d$, $d \geqslant 2$, is identified by its boundary spectral data (BSD). We establish that this result remains valid when finitely many data are removed. We also prove that the potential can be stably determined from knowledge of the asymptotics of the BSD.

WARNING: This is still a draft (hence the numerous typos) that will be regularly updated.

Contents

1. A short introduction to inverse spectral problems

Let $\Omega \subset \mathbb{R}^d$, where $d \in \mathbb{N} := \{1, 2, \dots\}$, be a bounded domain with $C^{1,1}$ boundary $\partial\Omega$. In the particular case where $d = 1$, we set $\Omega := (0, 1)$. Given $q \in L^\infty(\Omega)$, real-valued, we perturb the Dirichlet Laplacian in $L^2(\Omega)$ by q, i.e. we consider the operator acting in $L^2(\Omega)$ as $-\Delta + q$, that is endowed with homogeneous Dirichlet boundary conditions.

We investigate the inverse problem of determining the operator $-\Delta + q$, that is of determining the perturbation potential q, from knowledge of partial spectral data of $-\Delta + q$. More precisely, we are interested in two types of results:

- A *uniqueness result*, expressing that every two admissible potentials q_j, $j = 1, 2$, are equal whenever the spectral data of $-\Delta + q_1$ coincide with the ones of $-\Delta + q_2$, i.e.

$$(\text{Spectral data of } -\Delta + q_1 = \text{Spectral data of } -\Delta + q_2) \Longrightarrow (q_1 = q_2).$$

- A *stability result*, claiming that any unknown admissible potential q is not only uniquely determined (in the sense of the above implication) by the spectral data of $-\Delta+q$, but also that it depends continuously on these data.

1.1. **Self-adjointness, spectral data and all that.** For $q \in L^\infty(\Omega,\mathbb{R})$, we define A_q as the operator in $L^2(\Omega)$, associated with the closed sesquilinear form

$$a_q(u,v) := \int_\Omega \left(\nabla u(x) \cdot \overline{\nabla v(x)} + q(x)u(x)\overline{v(x)} \right) dx,\ u,v \in D(a_q) := H_0^1(\Omega), \tag{1.1}$$

where $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$, the set of infinitely differentiable and compactly supported functions in Ω, for the topology of the first-order Sobolev space $H^1(\Omega)$. The operator A_q is self-adjoint in $L^2(\Omega)$ and acts on its domain[1] as

$$A_q u = (-\Delta+q)u,\ u \in D(A_q) = H_0^1(\Omega) \cap H^2(\Omega). \tag{1.2}$$

Here, the notation $H^2(\Omega)$ stands for the usual second-order Sobolev space in Ω, and we recall that the graph norm of A_q is equivalent to the one of $H^2(\Omega)$, i.e.

$$c^{-1} \|u\|_{H^2(\Omega)} \leqslant \|u\|_{D(A_q)} := \|u\|_{L^2(\Omega)} + \|A_q u\|_{L^2(\Omega)} \leqslant c \|u\|_{H^2(\Omega)},\ u \in D(A_q), \tag{1.3}$$

for some constant $c \in (1,+\infty)$ that depends only on Ω and $\|q\|_{L^\infty(\Omega)}$.

Next, since the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, then the same is true for the resolvent of A_q, and the spectrum of A_q is discrete. We denote by $\{\lambda_n,\ n \in \mathbb{N}\}$ the non-decreasing sequence of eigenvalues of A_q, repeated with their multiplicity,

$$\lambda_1 \leqslant \ldots \leqslant \lambda_n \leqslant \lambda_{n+1} \leqslant \ldots,\ n \in \mathbb{N}.$$

In view of (1.1), we infer from the Min-Max principle that

$$\lambda_1 \geqslant -\|q\|_{L^\infty(\Omega)},$$

and we recall for further use that

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

Let $\{\varphi_n,\ n \in \mathbb{N}\}$ be an orthonormal basis in $L^2(\Omega)$ of eigenfunctions of A_q, such that

$$A_q\varphi_n = \lambda_n\varphi_n,\ n \in \mathbb{N}.$$

With reference to (1.3), we have

$$c^{-1}(1+\lambda_n) \leqslant \|\varphi_n\|_{H^2(\Omega)} \leqslant c(1+\lambda_n),\ n \in \mathbb{N}. \tag{1.4}$$

Put

$$\psi_n := \partial_\nu \varphi_n,\ n \in \mathbb{N},$$

where ν denotes the outward normal vector to $\partial\Omega$ and $\partial_\nu u := \nabla u \cdot \nu$ is the normal derivative of φ. Then, it follows from (1.4) and the continuity of the trace operator $\tau_1 : u \mapsto (\partial_\nu u)_{|\partial\Omega}$ from $H^2(\Omega)$ into $H^{1/2}(\partial\Omega)$, that

$$\|\psi_n\|_{H^{1/2}(\partial\Omega)} \leqslant \tilde{c}(1+\lambda_n),\ n \in \mathbb{N}, \tag{1.5}$$

for some positive constant $\tilde{c}$ that depends only on Ω and $\|q\|_{L^\infty(\Omega)}$.

[1] The assumption $\partial\Omega \in C^{1,1}$ is needed for applying the classical elliptic regularity theory, that establishes that $D(A_q) \subset H^2(\Omega)$.

1.2. **Review of the one-dimensional case.** Fix $d = 1$ and recall that we have $\Omega = (0, 1)$ with

$$A_q = -\frac{d^2}{dx^2} + q(x),\ D(A_q) = \{u \in H^2(0,1),\ u(0) = u(1) = 0\},$$

in this case.

1.2.1. *An obstruction to identifiability.* A very natural question that arises in this context is to know whether q can be determined by knowledge of $\mathrm{Sp}(A_q) = \{\lambda_n,\ n \in \mathbb{N}\}$. But the answer is negative as the spectrum does not discriminate between symmetric potentials. This can be seen by noticing that we have

$$UA_qU^{-1} = A_{Uq}, \tag{1.6}$$

where we have set $(Uf)(x) := f(1-x)$ for all $f \in L^2(\Omega)$ and a.e. $x \in \Omega$. Since U is unitary in $L^2(\Omega)$, then the operators A_q and A_{Uq} are unitarily equivalent, by (1.6). Hence they are isospectral: $\mathrm{Sp}(A_{Uq}) = \mathrm{Sp}(A_q)$. Thus, one cannot distinguish between the potentials q and Uq, from knowledge of the two spectra $\mathrm{Sp}(A_q)$ and $\mathrm{Sp}(A_{Uq})$, despite of the fact that $q \neq Uq$ when q is not symmetric about the midpoint $x = 1/2$ of the interval Ω.

Therefore, the spectrum of A_q does not uniquely determine q, and some additional spectral data is needed for identifying the potential.

1.2.2. *One-dimensional Borg-Levinson theorem.* Assuming that $\varphi_n'(0) = \frac{d\varphi_n}{dx}(0) = 1$ for all $n \in \mathbb{N}$, G. Borg [1] and N. Levinson [6] established when $\mathrm{Sp}(A_q)$ is known, that additional knowledge of $\{\|\varphi_n\|_{L^2(\Omega)},\ n \in \mathbb{N}\}$ uniquely determines q.

Theorem 1.1 (Borg (1946) and Levinson (1949))**.** *For $\lambda \in \mathbb{R}$ and for $q_j \in L^\infty(0,1;\mathbb{R})$, $j = 1, 2$, let $u_j(\cdot, \lambda)$ be the $H^2(0,1)$-solution to the initial values problem*

$$\begin{cases} (-\frac{d^2}{dx^2} + q_j(x))u_j(x,\lambda) = \lambda u_j(x,\lambda), & x \in (0,1) \\ u_j(0,\lambda) = 0,\ u_j'(0,\lambda) = 1. \end{cases}$$

Denote by $\{\lambda_{j,n},\ n \in \mathbb{N}\}$ the non-decreasing sequence of the Dirichlet eigenvalues associated with A_{q_j}, obtained by imposing:

$$u_j(1, \lambda_{j,n}) = 0,\ n \in \mathbb{N}.$$

Then, we have the implication:

$$\Big(\lambda_{1,n} = \lambda_{2,n}\ \text{and}\ \|u_1(\cdot,\lambda_{1,n})\|_{L^2(\Omega)} = \|u_2(\cdot,\lambda_{2,n})\|_{L^2(0,1)},\ n \in \mathbb{N}\Big) \Longrightarrow (q_1 = q_2\ \text{in}\ (0,1)).$$

Later on, I. M. Gel'fand and B. M. Levitan proved that uniqueness is still valid upon substituting $\varphi_n'(1)$ for $\|\varphi_n\|_{L^2(0,1)}$ in Theorem 1.1.

Theorem 1.2 (Gel'fand-Levitan (1951))**.** *Under the conditions of Theorem 1.1 we have:*

$$(\lambda_{1,n} = \lambda_{2,n}\ \text{and}\ u_1'(1,\lambda_{1,n}) = u_2'(1,\lambda_{2,n}),\ n \in \mathbb{N}) \Longrightarrow (q_1 = q_2\ \text{in}\ (0,1)).$$

This result was extended to higher dimensions $d \geqslant 2$ by several authors.

1.3. **Multidimensional results.**

1.3.1. *Boundary spectral data.* Let us recall that $\{\lambda_n,\ n \in \mathbb{N}\}$ is the non-decreasing sequence of the eigenvalues of A_q (repeated with the multiplicity), that $\{\varphi_n,\ n \in \mathbb{N}\}$ is a $L^2(\Omega)$-orthonormal basis of eigenfunctions of A_q such that $A_q \varphi_n = \lambda_n \varphi_n$, and that $\psi_n = \partial_\nu \varphi_n$. We define the boundary spectral data (BSD) of A_q, or the BSD associated with q, as:

$$\mathrm{BSD}(q) := \{(\lambda_n, \psi_n),\ n \in \mathbb{N}\}.$$

Remark 1.3. For all $n \in \mathbb{N}$, one may replace φ_n by $e^{i\theta_n}\varphi_n$ with $\theta_n \in \mathbb{R}$, in the above definition. Thus it is clear that the BSD are not defined in a unique way: they depend on the choice of the $L^2(\Omega)$-orthonormal basis $\{\varphi_n,\ n \in \mathbb{N}\}$ of eigenfunctions of A_q.

1.3.2. *Multidimensional results.* In 1988, it was proved for $d \geqslant 2$ by A. Nachman, J. Sylvester and G. Uhlmann in [7], and by R. Novikov in [8], that the potential q is uniquely determined by $\mathrm{BSD}(q)$, i.e. that

$$(\mathrm{BSD}(q_1) = \mathrm{BSD}(q_2)) \Longrightarrow (q_1 = q_2),$$

for any two suitable potentials q_j, $j = 1, 2$. This result has been improved in several ways by various authors.

Firstly, H. Isozaki [5] (see also M. Choulli [2]) extended the result of [7, 8] when finitely many eigenpairs remain unknown.

Theorem 1.4. *For $j = 1, 2$, let $q_j \in L^\infty(\Omega, \mathbb{R})$ and write*[2] $\mathrm{BSD}(q_j) = \{(\lambda_{j,n}, \psi_{j,n}),\ n \in \mathbb{N}\}$*. Then, for all $N \in \mathbb{N}$, we have the following implication:*

$$((\lambda_{1,n}, \psi_{1,n}) = (\lambda_{2,n}, \psi_{2,n}),\ n \geqslant N) \Longrightarrow (q_1 = q_2).$$

Recently, uniqueness in the determination of q was proved in [3, **?**] from the knowledge of the asymptotics of $\mathrm{BSD}(q)$ when $n \to +\infty$.

Theorem 1.5. *Let q_j for $j = 1, 2$, and the notations, be the same as in Theorem 1.4. Assume that the asymptotics of* $\mathrm{BSD}(q_1)$ *and* $\mathrm{BSD}(q_2)$ *coincide, in the sense that*

$$\lim_{n\to\infty} (\lambda_{1,n} - \lambda_{2,n}) = 0 \text{ and } \|\psi_{1,n} - \psi_{2,n}\|^2_{L^2(\partial\Omega)} < \infty.$$

Then, we have $q_1 = q_2$ in Ω.

2. Multidimensional inverse spectral theory

This section is devoted to the proof of Theorems 1.4 and 1.5. We start by establishing several technical results that are needed by the derivation of Theorems 1.4 and 1.5.

2.1. **Preliminaries.** For $q \in L^\infty(\Omega, \mathbb{R})$, $f \in H^{3/2}(\partial\Omega)$ and $\lambda \in \mathbb{C}$, we consider the boundary value problem (BVP)

$$\left\{ \begin{array}{ll} (-\Delta + q - \lambda)u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{array} \right. \tag{2.1}$$

First, we express the solution to (2.1) in terms of $\mathrm{BSD}(q)$.

[2]That is to say that $\{\lambda_{j,n},\ n \in \mathbb{N}\}$ is the non-decreasing sequence of the eigenvalues of A_{q_j} and that $\psi_{j,n} = \partial_\nu \varphi_{j,n}$ for all $n \in \mathbb{N}$, where $\{\varphi_{j,n},\ n \in \mathbb{N}\}$ is a $L^2(\Omega)$-orthonormal basis of eigenvectors of A_{q_j}, such that $A_{q_j}\varphi_{j,n} = \lambda_{j,n}\varphi_{j,n}$.

Lemma 2.1. *Let $q \in L^\infty(\Omega, \mathbb{R})$ and $f \in H^{3/2}(\partial\Omega)$. Then, for each $\lambda \in \mathbb{C} \setminus \mathrm{Sp}(A_q)$ there exists a unique solution $u_\lambda \in H^2(\Omega)$ to* (2.1)*. Moreover, we have*

$$u_\lambda = \sum_{n=1}^{+\infty} \frac{\langle f, \psi_n \rangle_{L^2(\partial\Omega)}}{\lambda - \lambda_n} \varphi_n \text{ in } L^2(\Omega), \tag{2.2}$$

and

$$\lim_{\lambda \to -\infty} \|u_\lambda\|^2_{L^2(\Omega)} = \lim_{\lambda \to -\infty} \left(\sum_{n=1}^{+\infty} \left| \frac{\langle f, \psi_n \rangle_{L^2(\partial\Omega)}}{\lambda - \lambda_n} \right|^2 \right) = 0. \tag{2.3}$$

Proof. We split the proof into three steps.

Step 1: Existence and uniqueness of the solution to (2.1). Since $f \in H^{3/2}(\partial\Omega)$ and since the trace operator $\tau_0 : v \mapsto v_{|\partial\Omega}$ is surjective from $H^2(\Omega)$ onto $H^{3/2}(\partial\Omega)$, then there exists $F \in H^2(\Omega)$ such that $\tau_0 F = f$. Thus, u_λ is a solution to (2.1) iff $v_\lambda := u_\lambda - F$ solves

$$\begin{cases} (-\Delta + q - \lambda)v = G & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

with $G := -(-\Delta + q - \lambda)F \in L^2(\Omega)$. Next, λ being in the resolvent set of A_q, we see that (2.4) admits a unique solution $v_\lambda = (A_q - \lambda)^{-1} G$. Thereofore, $u_\lambda = v_\lambda + F$ is the unique solution to (2.1).

Step 2: Proof of (2.2). For all $n \in \mathbb{N}$, we have

$$0 = \langle (-\Delta + q - \lambda) u_\lambda, \varphi_n \rangle_{L^2(\Omega)} = \int_\Omega (-\Delta + q(x) - \lambda) u_\lambda(x) \overline{\varphi_n(x)} dx,$$

whence

$$\begin{aligned} 0 &= -\int_{\partial\Omega} \partial_\nu u_\lambda(x) \overline{\varphi_n(x)} dx + \int_{\partial\Omega} u_\lambda(x) \overline{\psi_n(x)} dx + \int_\Omega u_\lambda(x) \overline{(A_q - \overline{\lambda}) \varphi_n(x)} dx \\ &= \int_{\partial\Omega} f(x) \overline{\psi_n(x)} dx + (\lambda_n - \lambda) \int_\Omega u_\lambda(x) \overline{\varphi_n(x)} dx, \end{aligned}$$

by integrating by parts. As a consequence we have $\langle u_\lambda, \varphi_n \rangle_{L^2(\Omega)} = \frac{\langle f, \psi_n \rangle_{L^2(\partial\Omega)}}{\lambda - \lambda_n}$, so (2.2) follows readily from this and the $L^2(\Omega)$-decomposition $u_\lambda = \sum_{n=1}^{+\infty} \langle u_\lambda, \varphi_n \rangle_{L^2(\Omega)} \varphi_n$.

Step 3: Proof of (2.3). With reference to (1.1), we have for all $n \in \mathbb{N}$,

$$\lambda_n = \langle A_q \varphi_n, \varphi_n \rangle_{L^2(\Omega)} = a_q(\varphi_n, \varphi_n) = \int_\Omega |\nabla \varphi_n(x)|^2 dx + \int_\Omega q(x) |\varphi_n(x)|^2 dx \geqslant -\|q\|_{L^\infty(\Omega)},$$

hence $\mathrm{Sp}(A_q) \subset [-\|q\|_{L^\infty(\Omega)}, +\infty)$. Thus, putting $\mu_0 := 1 + \|q\|_{L^\infty(\Omega)}$, we see that every $\lambda \in (-\infty, -\mu_0]$ lies in the resolvent set of A_q, and that

$$\left| \frac{\langle f, \psi_n \rangle_{L^2(\partial\Omega)}}{\lambda - \lambda_n} \right|^2 \leqslant \left| \frac{\langle f, \psi_n \rangle_{L^2(\partial\Omega)}}{\mu_0 + \lambda_n} \right|^2, \ n \in \mathbb{N}. \tag{2.5}$$

Further, since $\sum_{n=1}^{+\infty} \left| \frac{\langle f, \psi_n \rangle_{L^2(\partial\Omega)}}{\mu_0 + \lambda_n} \right|^2 = \|u_{-\mu_0}\|^2_{L^2(\Omega)} < \infty$, by (2.2) and the Parseval theorem, and since $\lim_{\lambda \to -\infty} \left| \frac{\langle f, \psi_n \rangle_{L^2(\partial\Omega)}}{\lambda - \lambda_n} \right|^2 = 0$ for all $n \in \mathbb{N}$, we infer from (2.5) and the Lebesgue dominated convergence theorem that

$$\lim_{\lambda\to-\infty}\left(\sum_{n=1}^{+\infty}\left|\frac{\langle f,\psi_n\rangle_{L^2(\partial\Omega)}}{\lambda-\lambda_n}\right|^2\right)=\sum_{n=1}^{+\infty}\left(\lim_{\lambda\to-\infty}\left|\frac{\langle f,\psi_n\rangle_{L^2(\partial\Omega)}}{\lambda-\lambda_n}\right|^2\right)=0.$$

Putting this, together with (2.2) and the Parseval formula, we obtain (2.3). □

Armed with Lemma 2.1, we may now quantify the influence of the spectral parameter λ, on the normal derivative of the solution to (2.1).

Lemma 2.2. *Let q and f be the same as in Lemma 2.1. Then, for all λ and μ in $\mathbb{C}\setminus\mathrm{Sp}(A_q)$, we have*

$$\partial_\nu(u_\lambda-u_\mu)=(\mu-\lambda)\sum_{n=1}^{+\infty}\frac{\langle f,\psi_n\rangle_{L^2(\partial\Omega)}}{(\lambda-\lambda_n)(\mu-\lambda_n)}\psi_n \text{ in } H^{1/2}(\partial\Omega). \tag{2.6}$$

Here u_λ (resp., u_μ) denotes the $H^2(\Omega)$-solution to (2.1) *(resp.,* (2.1) *where λ is replaced by μ), given by Lemma 2.1.*

Proof. In view of (2.1), we see that $v:=u_\lambda-u_\mu$ solves

$$\begin{cases}(-\Delta+q-\lambda)v=(\lambda-\mu)u_\mu & \text{in } \Omega\\ v=0 & \text{on } \partial\Omega.\end{cases} \tag{2.7}$$

Since λ is in the resolvent set of A_q, this entails that

$$v=(\lambda-\mu)(A_q-\lambda)^{-1}u_\mu=(\lambda-\mu)\sum_{n=1}^{+\infty}\frac{\langle u_\mu,\varphi_n\rangle_{L^2(\Omega)}}{\lambda_n-\lambda}\varphi_n, \tag{2.8}$$

the series being convergent in $L^2(\Omega)$. Recall that upon substituting μ for λ in (2.2), we have

$$u_\mu=\sum_{n=1}^{+\infty}\frac{\langle f,\psi_n\rangle_{L^2(\partial\Omega)}}{\mu-\lambda_n}\varphi_n \text{ in } L^2(\Omega). \tag{2.9}$$

Putting this together with (2.8), we get that

$$v=(\lambda-\mu)\sum_{n=1}^{+\infty}\frac{\langle f,\psi_n\rangle_{L^2(\partial\Omega)}}{(\lambda_n-\lambda)(\mu-\lambda_n)}\varphi_n \text{ in } L^2(\Omega), \tag{2.10}$$

Next, since $v\in D(A_q)$ and $A_qv=(\lambda-\mu)u_\mu+\lambda v$, we deduce from (2.9)-(2.10) that

$$A_qv=(\lambda-\mu)\sum_{n=1}^{+\infty}\frac{\lambda_n\langle f,\psi_n\rangle_{L^2(\partial\Omega)}}{(\lambda_n-\lambda)(\mu-\lambda_n)}\varphi_n \text{ in } L^2(\Omega).$$

Therefore, the series in (2.10) converges for the topology of the norm of A_q, hence it converges in $H^2(\Omega)$, according to (1.3). Finally, we obtain (2.6) from this upon invoking the continuity of the trace operator $\tau_1: u\mapsto(\partial_\nu u)_{|\partial\Omega}$ from $H^2(\Omega)$ into $H^{1/2}(\partial\Omega)$. □

The next lemma claims for any two real-valued bounded potentials q_1 and q_2, that the solutions to (2.1) associated with either $q=q_1$ or $q=q_2$, are closed as $\lambda\to-\infty$: in some sense the influence of the potential is dimmed when the spectral parameter λ goes to $-\infty$.

Lemma 2.3. *Let $f\in H^{3/2}(\partial\Omega)$ and let $q_j\in L^\infty(\Omega,\mathbb{R})$, $j=1,2$. For $\lambda\in\mathbb{C}\setminus(\mathrm{Sp}(A_{q_1})\cup\mathrm{Sp}(A_{q_2}))$, let $u_{j,\lambda}$ be the solution to* (2.1) *where q_j is substituted for q, given by Lemma 2.1. Then, we have*

$$\lim_{\lambda\to-\infty}\|\partial_\nu u_{1,\lambda}-\partial_\nu u_{2,\lambda}\|_{L^2(\partial\Omega)}=0. \tag{2.11}$$

Proof. Set $w_\lambda := u_{1,\lambda} - u_{2,\lambda}$, so we have

$$\left\{ \begin{array}{ll} (-\Delta + q_1 - \lambda) w_\lambda = (q_2 - q_1) u_{2,\lambda} & \text{in } \Omega \\ w_\lambda = 0 & \text{on } \partial\Omega, \end{array} \right.$$

from (2.1), and hence $w_\lambda = (A_{q_1} - \lambda)^{-1}(q_2 - q_1)u_{2,\lambda}$. Bearing in mind for all real number $\lambda < -\|q_1\|_{L^\infty(\Omega)}$, that $\|(A_{q_1} - \lambda)^{-1}\|_{\mathcal{B}(L^2(\Omega))} = \text{dist}^{-1}(\lambda, \text{Sp}(A_{q_1})) \leqslant 1/(\|q_1\|_{L^\infty(\Omega)} + \lambda)$, we find that

$$\|w_\lambda\|_{L^2(\Omega)} \leqslant \frac{\|q_2 - q_1\|_{L^\infty(\Omega)} \|u_{2,\lambda}\|_{L^2(\Omega)}}{-\lambda - \|q_1\|_{L^\infty(\Omega)}},\ \lambda \in \left(-\infty, -\|q_1\|_{L^\infty(\Omega)}\right).$$

From this and (2.3) it then follows that

$$\lim_{\lambda \to -\infty} \lambda \|w_\lambda\|_{L^2(\Omega)} = 0. \tag{2.12}$$

Next, since $A_{q_1} w_\lambda = (q_2 - q_1)u_{2,\lambda} + \lambda w_\lambda$, it holds true for every $\lambda < -\|q_1\|_{L^\infty(\Omega)}$ that

$$\|A_{q_1} w_\lambda\|_{L^2(\Omega)} \leqslant \|(q_2 - q_1)\|_{L^\infty(\Omega)} \|u_{2,\lambda}\|_{L^2(\Omega)} - \lambda \|w_\lambda\|_{L^2(\Omega)},$$

so we get $\lim_{\lambda \to -\infty} \|A_{q_1} w_\lambda\|_{L^2(\Omega)} = 0$, from (2.3) and (2.12). As a consequence we have

$$\lim_{\lambda \to -\infty} \left(\|w_\lambda\|_{L^2(\Omega)} + \|A_{q_1} w_\lambda\|_{L^2(\Omega)} \right) = 0.$$

This and (1.3) entail

$$\lim_{\lambda \to -\infty} \|u_{1,\lambda} - u_{2,\lambda}\|_{H^2(\Omega)} = 0$$

which together with the continuity of the trace operator $\tau_1 : u \mapsto (\partial_\nu u)_{|\partial\Omega}$ from $H^2(\Omega)$ into $H^{1/2}(\partial\Omega)$, yield (2.11). □

2.2. **Isozaki's asymptotic representation formula.** Let $q_j \in L^\infty(\Omega, \mathbb{R})$ satisfy

$$\|q_j\|_{L^\infty(\Omega)} \leqslant M,\ j = 1, 2, \tag{2.13}$$

for some *a priori* fixed constant $M \in (0, +\infty)$. In [5], H. Isozaki gives a simple representation formula, expressing the difference $q_1 - q_2$ in terms of the Dirichlet-to-Neumann (DN) operator associated with the BVP obtained by substituting q_j for q in (2.1). More precisely, adapting the argument of [5] to fit our aim in this text, we fix $\tau \in (1, +\infty)$ and we consider the BVP (2.1) with $\lambda = \lambda_\tau^+ := (\tau + i)^2$ and $q = q_j$, i.e.

$$\left\{ \begin{array}{ll} (-\Delta + q_j - \lambda_\tau^+) u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{array} \right. \tag{2.14}$$

Let u_{j,λ_τ^+} be the $H^2(\Omega)$-solution to (2.14) (for the sake of notational simplicity we drop the dependence of u_{j,λ_τ^+} on f). Let us introduce the DN map associated with (2.14), as

$$\begin{array}{rccc} \Lambda_{j,\lambda_\tau^+} : & H^{3/2}(\partial\Omega) & \to & H^{1/2}(\partial\Omega) \\ & f & \mapsto & \left(\partial_\nu u_{j,\lambda_\tau^+} \right)_{|\partial\Omega}. \end{array} \tag{2.15}$$

Given two test functions $f_\tau^\pm$, we shall make precise in the coming section, we aim to link the difference $q_1 - q_2$ to the asymptotic behavior of

$$S_\tau := S_{1,\tau} - S_{2,\tau},\ \text{where } S_{j,\tau} := \langle \Lambda_{j,\lambda_\tau^+} f_\tau^+, f_\tau^- \rangle_{L^2(\partial\Omega)}, \tag{2.16}$$

as $\tau \to +\infty$.

2.2.1. *Test functions.* For $\xi \in \mathbb{R}^d$ fixed, and for every $\tau \in (|\xi|+1, +\infty)$, we set $\lambda_\tau^\pm := (\tau \pm i)^2$, and we seek two functions $f_\tau^\pm$ such that

$$\left\{ \begin{array}{l} (-\Delta - \lambda_\tau^\pm) f_\tau^\pm = 0 \text{ in } \Omega \\ \lim_{\tau \to +\infty} f_\tau^+(x) \overline{f_\tau^-(x)} = e^{-i\xi \cdot x},\ x \in \Omega \\ \sup_{\tau \in (|\xi|+1, +\infty)} \|f_\tau^\pm\|_{L^\infty(\Omega)} < \infty. \end{array} \right. \tag{2.17}$$

Here and in the remaining part of this text, the notation $\cdot$ (resp., $|\cdot|$) stands for the Euclidian product (resp., norm) in $\mathbb{R}^d$.

Pick $\eta \in \mathbb{S}^{d-1}$ such that $\xi \cdot \eta = 0$, and put

$$\beta_\tau := \sqrt{1 - \frac{|\xi|^2}{4\tau^2}} \text{ and } \eta_\tau^\pm := \beta_\tau \eta \mp \frac{\xi}{2\tau}, \tag{2.18}$$

in such a way that $|\eta_\tau^\pm| = 1$. Then, the two following functions

$$f_\tau^\pm(x) := e^{i(\tau \pm i)\eta_\tau^\pm \cdot x},\ x \in \Omega, \tag{2.19}$$

fulfill the conditions of (2.17). As a matter of fact, it can be checked through direct computation from (2.18)-(2.19), that $\Delta f_\tau^\pm = -\lambda_\tau^\pm |\eta_\tau^\pm|^2 f_\tau^\pm = -\lambda_\tau^\pm f_\tau^\pm$ in Ω, that $f_\tau^+(x)\overline{f_\tau^-(x)} = e^{-i\frac{\tau+i}{\tau}\xi \cdot x}$ for all $x \in \Omega$, and that

$$|f_\tau^\pm(x)| \leqslant e^{|x|},\ x \in \overline{\Omega}. \tag{2.20}$$

We notice for further use from (2.20) that the estimate

$$\|f_\tau^\pm\|_{L^p(X)} \leqslant c_* := \left(1 + |\Omega|^{1/2} + |\partial\Omega|^{1/2}\right) \sup_{x \in \overline{\Omega}} e^{|x|}, \tag{2.21}$$

holds with $X = \Omega$ or $X = \partial\Omega$, and with $p = 2$ or $p = \infty$.

2.2.2. *Probing* (2.1) *with the test function* $f_\tau^\pm$. For $j = 1, 2$ and $z \in \mathbb{C} \setminus \mathrm{Sp}(A_{q_j})$, we denote by $u_{j,z}^\pm$ the $H^2(\Omega)$-solution to the BVP (2.1), where $(q_j, z, f_\tau^\pm)$ is substituted for (q, λ, f). The function $u_{j,z}^\pm$ is characterized by

$$\left\{ \begin{array}{ll} (-\Delta + q_j - z) u_{j,z}^\pm = 0 & \text{in } \Omega \\ u_{j,z}^\pm = f_\tau^\pm & \text{on } \partial\Omega, \end{array} \right. \tag{2.22}$$

hence $v_{j,z}^\pm := u_{j,z}^\pm - f_\tau^\pm$ solves

$$\left\{ \begin{array}{ll} (-\Delta + q_j - z) v_{j,z}^\pm = -(-\Delta + q_j - z) f_\tau^\pm & \text{in } \Omega \\ v_{j,z}^\pm = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.23}$$

Moreover, since $(-\Delta + q_j - z) f_\tau^\pm = (q_j + \lambda_\tau^\pm - z) f_\tau^\pm$, by the first line in (2.17), it follows from (2.23) that

$$v_{j,z}^\pm = -(A_{q_j} - z)^{-1} (q_j + \lambda_\tau^\pm - z) f_\tau^\pm. \tag{2.24}$$

Let us now examine the case where $z = \lambda_\tau^\pm$, which is permitted since $\lambda_\tau^\pm$ is in the resolvent set of the self-adjoint operator A_{q_j}, as we have:

$$\mathrm{Im}\,(\lambda_\tau^\pm) = \pm 2\tau \neq 0. \tag{2.25}$$

We shall establish that the $L^2(\Omega)$-norm of $v^{\pm}_{j,\lambda_\tau^\pm}$ scales like τ^{-1} as τ becomes large, whereas the one of $u^{\pm}_{j,\lambda_\tau^\pm}$ is bounded uniformly in $\tau \in (1+|\xi|,+\infty)$. To do that, we combine the first line of (2.18) with (2.24), where $\lambda_\tau^\pm$ is substituted for z, and get $v^{\pm}_{j,\lambda_\tau^\pm} = (A_{q_j} - \lambda_\tau^\pm)^{-1} q_j f_\tau^\pm$. Thus, using that

$$\left\| (A_{q_j} - \lambda_\tau^\pm)^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} = \frac{1}{\operatorname{dist}(\lambda_\tau^\pm, \operatorname{Sp}(A_{q_j}))} \leqslant \frac{1}{2\tau},\ j=1,2,$$

we obtain

$$\left\| v^{\pm}_{j,\lambda_\tau^\pm} \right\|_{L^2(\Omega)} \leqslant \frac{\|q_j\|_{L^\infty(\Omega)} \|f_\tau^\pm\|_{L^2(\Omega)}}{2\tau} \leqslant \frac{Mc_*}{2\tau},\ j=1,2, \tag{2.26}$$

upon applying (2.13) and (2.21) with $(X,p)=(\Omega,2)$. Now, bearing in mind that $\tau \geqslant 1$ and recalling that $u^\pm_{j,z} = v^\pm_{j,z} + f_\tau^\pm$, we derive from (2.26) that

$$\left\| u^{\pm}_{j,\lambda_\tau^\pm} \right\|_{L^2(\Omega)} \leqslant \frac{M+2}{2} c_*,\ j=1,2. \tag{2.27}$$

2.2.3. *Isozaki's formula.*

Proposition 2.4. *For $j=1,2$, let $q_j \in L^\infty(\Omega,\mathbb{R})$ satisfy* (2.13)*. Then, for all $\xi \in \mathbb{R}^d$, we have*

$$\int_\Omega (q_1(x) - q_2(x)) e^{-i\xi\cdot x} dx = \lim_{\tau \to +\infty} S_\tau, \tag{2.28}$$

where S_τ is defined by (2.15)-(2.16).

Proof. For $j=1,2$, we consider the $H^2(\Omega)$-solution u^+_{j,λ_τ^+} to the BVP (2.14) with $f=f_\tau^+$:

$$\left\{ \begin{array}{ll} (-\Delta + q_j - \lambda_\tau^+) u^+_{j,\lambda_\tau^+} = 0 & \text{in } \Omega \\ u^+_{j,\lambda_\tau^+} = f_\tau^+ & \text{on } \partial\Omega. \end{array} \right. \tag{2.29}$$

Upon left-multiplying the first line of (2.29) by $\overline{f_\tau^-}$, integrating on Ω, and applying the Green formula, we obtain with the help of (2.17) that

$$\begin{aligned} 0 &= \int_\Omega (-\Delta + q_j - \lambda_\tau^+) u^+_{j,\lambda_\tau^+}(x) \overline{f_\tau^-(x)} dx \\ &= \langle f_\tau^+, \partial_\nu f_\tau^- \rangle_{L^2(\partial\Omega)} - \langle \partial_\nu u^+_{j,\lambda_\tau^+}, f_\tau^- \rangle_{L^2(\partial\Omega)} + \int_\Omega u^+_{j,\lambda_\tau^+}(x) \overline{(-\Delta + q_j - \lambda_\tau^-) f_\tau^-(x)} dx \\ &= \langle f_\tau^+, \partial_\nu f_\tau^- \rangle_{L^2(\partial\Omega)} - S_{j,\tau} + \int_\Omega q_j(x) u^+_{j,\lambda_\tau^+}(x) \overline{f_\tau^-(x)} dx,\ j=1,2. \end{aligned}$$

Thus, for $j=1,2$, we have $S_{j,\tau} = \langle f_\tau^+, \partial_\nu f_\tau^- \rangle_{L^2(\partial\Omega)} + \int_\Omega q_j(x) u^+_{j,\lambda_\tau^+}(x) \overline{f_\tau^-(x)} dx$, and consequently

$$S_\tau = S_{1,\tau} - S_{2,\tau} = \int_\Omega \left(q_1(x) u^+_{1,\lambda_\tau^+}(x) - q_2(x) u^+_{2,\lambda_\tau^+}(x) \right) \overline{f_\tau^-(x)} dx. \tag{2.30}$$

Next, since $u^+_{j,\lambda_\tau^+} = v^+_{j,\lambda_\tau^+} + f_\tau^+$ for $j=1,2$, we see from (2.30) that

$$S_\tau - \int_\Omega (q_1(x) - q_2(x)) f_\tau^+(x) \overline{f_\tau^-(x)} dx = \int_\Omega (q_1(x) v^+_{1,\lambda_\tau^+}(x) - q_2(x) v^+_{2,\lambda_\tau^+}(x)) \overline{f_\tau^-(x)} dx.$$

Therefore, we get

$$\left| S_\tau - \int_\Omega (q_1(x) - q_2(x)) f_\tau^+(x) \overline{f_\tau^-(x)} dx \right| \leqslant \left(\sum_{j=1}^2 \|q_j\|_{L^\infty(\Omega)} \left\| v^+_{j,\lambda_\tau^+} \right\|_{L^2(\Omega)} \right) \|f_\tau^-\|_{L^2(\Omega)} \leqslant \frac{M^2 c_*^2}{\tau},$$

upon applying (2.21) with $(X,p)=(\Omega,2)$ and (2.26), and consequently:

$$\lim_{\tau\to+\infty}\left(S_\tau-\int_\Omega (q_1(x)-q_2(x))f_\tau^+(x)\overline{f_\tau^-(x)}dx\right)=0. \tag{2.31}$$

Finally, as we have

$$\lim_{\tau\to+\infty}\int_\Omega (q_1(x)-q_2(x))f_\tau^+(x)\overline{f_\tau^-(x)}dx=\int_\Omega (q_1(x)-q_2(x))e^{-i\xi\cdot x}dx,$$

by (2.17), (2.21) with $(X,p)=(\Omega,+\infty)$, (2.13), and the dominated convergence theorem, the desired result follows directly from this and from (2.31). □

2.3. **Proof of Theorem 1.4.** In view of Proposition 2.4, we are left with the task of proving that

$$\lim_{\tau\to+\infty}S_\tau=0,\ \xi\in\mathbb{R}^d. \tag{2.32}$$

Indeed, if we combine (2.32) with Isozaki's formula (2.28), we get that the Fourier transform

$$(\mathcal{F}q)(\xi):=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}q(x)e^{-i\xi\cdot x}dx \tag{2.33}$$

of the function

$$q(x):=\left\{\begin{array}{cl} q_1(x)-q_2(x) & \text{if } x\in\Omega\\ 0 & \text{if } x\in\mathbb{R}^d\setminus\Omega,\end{array}\right. \tag{2.34}$$

reads $(\mathcal{F}q)(\xi)=\frac{1}{(2\pi)^{d/2}}\int_\Omega (q_1(x)-q_2(x))e^{-i\xi\cdot x}dx=\frac{1}{(2\pi)^{d/2}}\lim_{\tau\to+\infty}S_\tau=0$ for every $\xi\in\mathbb{R}^d$. By the injectivity of Fourier transform $\mathcal{F}$, this entails that $q=0$ in $\mathbb{R}^d$, i.e. that $q_1=q_2$ in Ω.

We turn now to establishing (2.32). To this purpose, we fix $\xi\in\mathbb{R}^d\setminus\{0\}$, pick $\tau\in(|\xi|+1,+\infty)$, and for $j=1,2$ and all $z\in\mathbb{C}\setminus\mathrm{Sp}(A_{q_j})$, we consider the $H^2(\Omega)$-solution $u_{j,z}^+$ to the BVP (2.1), where (q_j,z,f_τ^+) is substituted for (q,λ,f), i.e.

$$\left\{\begin{array}{ll} (-\Delta+q_j-z)u_{j,z}^+=0 & \text{in } \Omega\\ u_{j,z}^+=f_\tau^+ & \text{on } \partial\Omega.\end{array}\right.$$

For $z_j\in\mathbb{C}\setminus\mathrm{Sp}(A_{q_j})$, $j=1,2$, we put $v_{j,z_1,z_2}^+:=u_{j,z_1}^+-u_{j,z_2}^+$ and recall (2.16). We get that

$$\begin{array}{rcl} S_\tau &=& \langle\Lambda_{1,\lambda_\tau^+}f_\tau^+,f_\tau^-\rangle_{L^2(\partial\Omega)}-\langle\Lambda_{2,\lambda_\tau^+}f_\tau^+,f_\tau^-\rangle_{L^2(\partial\Omega)}\\ &=& \langle\partial_\nu u_{1,\lambda_\tau^+}^+-\partial_\nu u_{2,\lambda_\tau^+}^+,f_\tau^-\rangle_{L^2(\partial\Omega)}\\ &=& \langle\partial_\nu v_{1,\lambda_\tau^+,\mu}^+,f_\tau^-\rangle_{L^2(\partial\Omega)}-\langle\partial_\nu v_{2,\lambda_\tau^+,\mu}^+,f_\tau^-\rangle_{L^2(\partial\Omega)}+\langle\partial_\nu u_{1,\mu}^+-\partial_\nu u_{2,\mu}^+,f_\tau^-\rangle_{L^2(\partial\Omega)}, \end{array} \tag{2.35}$$

for every $\mu\in(-\infty,-M)$. Thus, bearing in mind that $\lim_{\mu\to-\infty}\left\|\partial_\nu u_{1,\mu}^+-\partial_\nu u_{2,\mu}^+\right\|_{L^2(\partial\Omega)}=0$, from (2.11), we find upon sending μ to $-\infty$ in (2.35), that

$$S_\tau=\lim_{\mu\to-\infty}\langle\partial_\nu v_{1,\lambda_\tau^+,\mu}^+-\partial_\nu v_{2,\lambda_\tau^+,\mu}^+,f_\tau^-\rangle_{L^2(\partial\Omega)}. \tag{2.36}$$

Let us introduce

$$\kappa_{\tau,\mu}(t):=\frac{\mu-\lambda_\tau^+}{(\lambda_\tau^+-t)(\mu-t)},\ t\in\mathbb{R}\setminus\{\mu\}, \tag{2.37}$$

and

$$\zeta_\tau(\psi,\varphi):=\langle f_\tau^+,\psi\rangle_{L^2(\partial\Omega)}\overline{\langle f_\tau^-,\varphi\rangle}_{L^2(\partial\Omega)},\ \psi,\ \varphi\in L^2(\partial\Omega). \tag{2.38}$$

Then, we invoke Lemma 2.2 in order to decompose the scalar product in the right hand side of (2.36), as

$$\langle \partial_\nu v^+_{1,\lambda^+_\tau,\mu} - \partial_\nu v^+_{2,\lambda^+_\tau,\mu}, f^-_\tau \rangle_{L^2(\partial\Omega)} = \sum_{n=1}^{+\infty} \left(\kappa_{\tau,\mu}(\lambda_{1,n})\zeta_\tau(\psi_{1,n}) - \kappa_{\tau,\mu}(\lambda_{2,n})\zeta_\tau(\psi_{2,n}) \right), \tag{2.39}$$

where the notation $\zeta_\tau(\psi)$ is a shorthand for $\zeta_\tau(\psi,\psi)$. Next, we have $(\lambda_{1,n},\psi_{1,n}) = (\lambda_{2,n},\psi_{2,n})$ for all $n \geqslant N$, by assumption, hence (2.39) reads

$$\langle \partial_\nu v^+_{1,\lambda^+_\tau,\mu} - \partial_\nu v^+_{2,\lambda^+_\tau,\mu}, f^-_\tau \rangle_{L^2(\partial\Omega)} = \sum_{n=1}^{N-1} \left(\kappa_{\tau,\mu}(\lambda_{1,n})\zeta_\tau(\psi_{1,n}) - \kappa_{\tau,\mu}(\lambda_{2,n})\zeta_\tau(\psi_{2,n}) \right), \tag{2.40}$$

the sum being taken equal to zero when $N=1$. Further, taking into account that $\lim_{\mu\to-\infty} \kappa_{\tau,\mu}(\lambda_{j,n}) = 1/(\lambda^+_\tau - \lambda_{j,n})$ for all $j=1,2$ and all $n=1,\ldots,N-1$, we derive from (2.36) and (2.40), that

$$S_\tau = \sum_{n=1}^{N-1} \left(\frac{\zeta_\tau(\psi_{1,n})}{\lambda^+_\tau - \lambda_{1,n}} - \frac{\zeta_\tau(\psi_{2,n})}{\lambda^+_\tau - \lambda_{2,n}} \right).$$

Moreover, since $\mathrm{Im}\,(\lambda^+_\tau) = 2\tau$, we have $|\lambda^+_\tau - \lambda_{j,n}| \geqslant 2\tau$ for $j=1,2$ and for all $n=1,\ldots,N-1$, whence

$$\begin{aligned} |S_\tau| &\leqslant \sum_{n=1}^{N-1} \left(|\zeta_\tau(\psi_{1,n})| + |\zeta_\tau(\psi_{2,n})| \right) (2\tau)^{-1} \\ &\leqslant \left\| f^+_\tau \right\|_{L^2(\partial\Omega)} \| f^-_\tau \|_{L^2(\partial\Omega)} \sum_{n=1}^{N-1} \left(\|\psi_{1,n}\|^2_{L^2(\partial\Omega)} + \|\psi_{2,n}\|^2_{L^2(\partial\Omega)} \right) (2\tau)^{-1}. \end{aligned}$$

This entails that $|S_\tau| \leqslant c_*^2 \sum_{n=1}^{N-1} \left(\|\psi_{1,n}\|^2_{L^2(\partial\Omega)} + \|\psi_{2,n}\|^2_{L^2(\partial\Omega)} \right) \tau^{-1}$, upon applying (2.21) with $X = \partial\Omega$ and $p=2$, which gives (2.32).

2.4. **Proof of Theorem 1.5.** We stick with the notations of Section 2.3 and recall from (2.38)-(2.39) that

$$S_\tau = \lim_{\mu\to-\infty} \langle \partial_\nu v^+_{1,\lambda^+_\tau,\mu} - \partial_\nu v^+_{2,\lambda^+_\tau,\mu}, f^-_\tau \rangle_{L^2(\partial\Omega)} = \lim_{\mu\to-\infty} \sum_{n=1}^{+\infty} \left(A_{n,\tau,\mu} + B_{n,\tau,\mu} \right), \tag{2.41}$$

where

$$A_{n,\tau,\mu} := \left(\kappa_{\tau,\mu}(\lambda_{1,n}) - \kappa_{\tau,\mu}(\lambda_{2,n}) \right) \zeta_\tau(\psi_{1,n}), \tag{2.42}$$

and

$$B_{n,\tau,\mu} := \kappa_{\tau,\mu}(\lambda_{2,n}) \left(\zeta_\tau(\psi_{1,n} - \psi_{2,n}, \psi_{1,n}) + \zeta_\tau(\psi_{2,n}, \psi_{1,n} - \psi_{2,n}) \right). \tag{2.43}$$

We split the proof into three steps. The first one, presented in Section 2.4.1, is to show that

$$\lim_{\mu\to-\infty} \sum_{n=1}^{+\infty} A_{n,\tau,\mu} = \sum_{n=1}^{+\infty} A_{n,\tau,*}, \text{ where } A_{n,\tau,*} := \frac{(\lambda_{1,n} - \lambda_{2,n})\zeta_\tau(\psi_{1,n})}{(\lambda^+_\tau - \lambda_{1,n})(\lambda^+_\tau - \lambda_{2,n})}, \tag{2.44}$$

while the second one is to establish in Section 2.4.2 that

$$\lim_{\mu\to-\infty} \sum_{n=1}^{+\infty} B_{n,\tau,\mu} = \sum_{n=1}^{+\infty} B_{n,\tau,*}, \text{ where } B_{n,\tau,*} := \frac{\zeta_\tau(\psi_{1,n} - \psi_{2,n}, \psi_{1,n}) + \zeta_\tau(\psi_{2,n}, \psi_{1,n} - \psi_{2,n})}{\lambda^+_\tau - \lambda_{2,n}}. \tag{2.45}$$

Finally, Section 2.4.3 contains the end of the proof.

2.4.1. *Step 1: Proof of* (2.44). Let us start by noticing that

$$|\kappa_{\tau,\mu}(\lambda_{1,n})-\kappa_{\tau,\mu}(\lambda_{2,n})| \leqslant 2\,|\lambda_{1,n}-\lambda_{2,n}| \max_{t\in[\lambda_{1,n},\lambda_{2,n}]}\left(\frac{1}{\left|\lambda_\tau^+-t\right|^2}+\frac{1}{|\mu-t|^2}\right),\ n\in\mathbb{N}.$$

This can be seen from the identity $\kappa_{\tau,\mu}(\lambda_{1,n})-\kappa_{\tau,\mu}(\lambda_{2,n})=\int_{\lambda_{1,n}}^{\lambda_{2,n}}\kappa_{\tau,\mu}'(t)dt$, which yields

$$|\kappa_{\tau,\mu}(\lambda_{1,n})-\kappa_{\tau,\mu}(\lambda_{2,n})| \leqslant |\lambda_{1,n}-\lambda_{2,n}| \max_{t\in[\lambda_{1,n},\lambda_{2,n}]}\left(\frac{|\lambda_\tau^+-\mu|}{\left|\lambda_\tau^+-t\right||\mu-t|^2}+\frac{|\lambda_\tau^+-\mu|}{\left|\lambda_\tau^+-t\right|^2|\mu-t|}\right),$$

and from the basic estimate $|\lambda_\tau^+-\mu|\leqslant|\lambda_\tau^+-t|+|\mu-t|$, entailing:

$$\begin{aligned}\frac{|\lambda_\tau^+-\mu|}{\left|\lambda_\tau^+-t\right||\mu-t|^2}+\frac{|\lambda_\tau^+-\mu|}{\left|\lambda_\tau^+-t\right|^2|\mu-t|} &\leqslant \frac{1}{|\mu-t|^2}+\frac{2}{\left|\lambda_\tau^+-t\right||\mu-t|}+\frac{1}{\left|\lambda_\tau^+-t\right|^2}\\ &\leqslant \frac{2}{|\mu-t|^2}+\frac{2}{\left|\lambda_\tau^+-t\right|^2}.\end{aligned}$$

Denote by $\lambda_{*,n}$ a real number between $\lambda_{1,n}$ and $\lambda_{2,n}$, where the maximum of the function $t\mapsto \left|\lambda_\tau^+-t\right|^{-2}+|\mu-t|^{-2}$ is achieved, in such a way that

$$|\kappa_{\tau,\mu}(\lambda_{1,n})-\kappa_{\tau,\mu}(\lambda_{2,n})| \leqslant 2\,|\lambda_{1,n}-\lambda_{2,n}|\left(\frac{1}{\left|\lambda_\tau^+-\lambda_{*,n}\right|^2}+\frac{1}{|\mu-\lambda_{*,n}|^2}\right),\ n\in\mathbb{N}. \tag{2.46}$$

Next, bearing in mind that $\lim_{n\to+\infty}\lambda_{1,n}=+\infty$, we pick $N_0\in\mathbb{N}$ so large, that

$$\lambda_{1,N_0}\geqslant\left|\lambda_\tau^+\right|+4M. \tag{2.47}$$

Since $\lambda_{1,n}\geqslant\lambda_{1,N_0}$ for all $n\geqslant N_0$, we have $|\lambda_\tau^+-\lambda_{1,n}|\geqslant\lambda_{1,n}-|\lambda_\tau^+|\geqslant 4M$ in this case, whence $|\lambda_\tau^+-\lambda_{*,n}|\geqslant|\lambda_\tau^+-\lambda_{1,n}|-|\lambda_{1,n}-\lambda_{*,n}|\geqslant|\lambda_\tau^+-\lambda_{1,n}|-2M$. Here, we used the basic inequality $|\lambda_{1,n}-\lambda_{*,n}|\leqslant|\lambda_{1,n}-\lambda_{2,n}|$ and the estimate

$$|\lambda_{1,n}-\lambda_{2,n}|\leqslant\|q_1-q_2\|_{L^\infty(\Omega)}\leqslant 2M,\ n\in\mathbb{N}, \tag{2.48}$$

arising from the Min-Max principle and the operator identity $A_{q_2}=A_{q_1}+q_2-q_1$. Therefore, we find

$$\left|\lambda_\tau^+-\lambda_{*,n}\right|\geqslant\frac{|\lambda_\tau^+-\lambda_{1,n}|}{2},\ n\geqslant N_0. \tag{2.49}$$

Similarly, taking $\mu\in(-\infty,-(1+5M)]$, we have $|\mu-\lambda_{1,n}|\geqslant-\mu-M\geqslant 4M$. Since $|\lambda_{1,n}-\lambda_{*,n}|\leqslant 2M$, from (2.48), then we have $|\lambda_{1,n}-\lambda_{*,n}|\leqslant|\mu-\lambda_{1,n}|/2$, and hence

$$|\mu-\lambda_{*,n}|\geqslant|\mu-\lambda_{1,n}|-|\lambda_{1,n}-\lambda_{*,n}|\geqslant\frac{|\mu-\lambda_{1,n}|}{2},\ n\in\mathbb{N}.$$

Putting this together with (2.42), (2.46) and (2.49), we obtain

$$|A_{n,\tau,\mu}|\leqslant 8\delta_1\left(\frac{|\zeta_\tau(\psi_{1,n})|}{\left|\lambda_\tau^+-\lambda_{1,n}\right|^2}+\frac{|\zeta_\tau(\psi_{1,n})|}{|\mu-\lambda_{1,n}|^2}\right),\ n\geqslant N_0, \tag{2.50}$$

where $\delta_1:=\sup_{n\in\mathbb{N}}|\lambda_{1,n}-\lambda_{2,n}|<\infty$.

Further, with reference to (2.38), an application of the Cauchy-Schwarz inequality and (2.2) yields

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{|\zeta_\tau(\psi_{1,n})|}{|\ell-\lambda_{1,n}|^2} &\leqslant \left(\sum_{n=1}^{+\infty}\left|\frac{\langle f_\tau^+,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\ell-\lambda_{1,n}}\right|^2\right)^{1/2}\left(\sum_{n=1}^{+\infty}\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\overline{\ell}-\lambda_{1,n}}\right|^2\right)^{1/2}\\ &\leqslant \left\|u_{1,\ell}^+\right\|_{L^2(\Omega)}\left\|u_{1,\overline{\ell}}^-\right\|_{L^2(\Omega)},\ \ell=\lambda_\tau^+,\ \mu.\end{aligned}$$

Here $u_{1,\ell}^\pm$ denotes the $H^2(\Omega)$-solution to (2.1) where $(\ell,q_1,f_\tau^\pm)$ is substituted for (λ,q,f). Thus, bearing in mind that $\overline{\lambda_\tau^+}=\lambda_\tau^-$, we derive from (2.50) that

$$\sum_{n=N_0}^{+\infty}|A_{n,\tau,\mu}|\leqslant 8\delta_1\left(\left\|u_{1,\lambda_\tau^+}^+\right\|_{L^2(\Omega)}\left\|u_{1,\lambda_\tau^-}^-\right\|_{L^2(\Omega)}+\left\|u_{1,\mu}^+\right\|_{L^2(\Omega)}\left\|u_{1,\mu}^-\right\|_{L^2(\Omega)}\right).$$

With reference to (2.3), we may assume (upon possibly enlarging $-\mu$) that $\left\|u_{1,\mu}^\pm\right\|_{L^2(\Omega)}\leqslant 1$, so we obtain

$$\sum_{n=N_0}^{+\infty}|A_{n,\tau,\mu}|\leqslant 2\delta_1\left((M+2)^2c_*^2+4\right), \tag{2.51}$$

with the aid of (2.27). Now, bearing in mind that $\lim_{\mu\to-\infty}A_{n,\tau,\mu}=A_{n,\tau,*}$ for all $n\in\mathbb{N}$, we deduce (2.44) from this and (2.51) by invoking the Lebesgue dominated convergence theorem.

2.4.2. *Step 2: Proof of* (2.45). For all $n\geqslant N_0$, we infer from (2.47)-(2.48) that

$$\lambda_{2,n}\geqslant\lambda_{1,n}-|\lambda_{1,n}-\lambda_{2,n}|\geqslant\lambda_{1,N_0}-2M\geqslant\left|\lambda_\tau^+\right|,$$

whence $|\mu-\lambda_{2,n}|=\lambda_{2,n}-\mu\geqslant|\lambda_\tau^+|-\mu\geqslant|\lambda_\tau^+-\mu|$. Therefore, in view of (2.37), we get

$$|\kappa_{\tau,\mu}(\lambda_{2,n})|\leqslant\frac{1}{\left|\lambda_\tau^+-\lambda_{2,n}\right|},\ n\geqslant N_0,$$

which combined with (2.38), entails

$$\begin{aligned}|B_{n,\tau,\mu}| &\leqslant \|\psi_{1,n}-\psi_{2,n}\|_{L^2(\partial\Omega)}\left(\left\|f_\tau^+\right\|_{L^2(\partial\Omega)}\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|+\|f_\tau^-\|_{L^2(\partial\Omega)}\left|\frac{\langle f_\tau^+,\psi_{2,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|\right)\\ &\leqslant c_*\|\psi_{1,n}-\psi_{2,n}\|_{L^2(\partial\Omega)}\left(\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|+\left|\frac{\langle f_\tau^+,\psi_{2,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|\right).\end{aligned} \tag{2.52}$$

In the last line we applied (2.21) with $(X,p)=(\partial\Omega,2)$.

Further, recalling from (2.47) that $|\lambda_\tau^+-\lambda_{1,n}|\geqslant 2\|q_1-q_2\|_{L^\infty(\Omega)}$ for all $n\geqslant N_0$, and using the estimate $|\lambda_\tau^+-\lambda_{2,n}|\geqslant|\lambda_\tau^+-\lambda_{1,n}|-\|q_1-q_2\|_{L^\infty(\Omega)}$ arising from (2.48), we find that

$$\left|\lambda_\tau^+-\lambda_{2,n}\right|\geqslant\frac{|\lambda_\tau^+-\lambda_{1,n}|}{2}=\frac{|\lambda_\tau^--\lambda_{1,n}|}{2},\ n\geqslant N_0. \tag{2.53}$$

Putting this together with (2.52), we get by applying the Cauchy-Schwarz inequality, that the following estimate

$$\sum_{n=N_0}^{+\infty}|B_{n,\tau,\mu}|\leqslant 2^{3/2}c_*\varepsilon_1\left(\left(\sum_{n=N_0}^{+\infty}\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^--\lambda_{1,n}}\right|^2\right)^{1/2}+\left(\sum_{n=N_0}^{+\infty}\left|\frac{\langle f_\tau^+,\psi_{2,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|^2\right)^{1/2}\right),$$

holds with $\varepsilon_1 := \left(\sum_{n=1}^{+\infty} \|\psi_{1,n} - \psi_{2,n}\|_{L^2(\partial\Omega)}^2 \right)^{1/2}$, for every $\mu \in (-\infty, -(1+5M)]$. This leads to

$$\sum_{n=N_0}^{+\infty} |B_{n,\tau,\mu}| \leqslant 2^{3/2} c_* \varepsilon_1 \left(\left\| u_{1,\lambda_\tau^-}^- \right\|_{L^2(\Omega)} + \left\| u_{2,\lambda_\tau^+}^+ \right\|_{L^2(\Omega)} \right) \leqslant 2^{7/2}(M+2) c_*^2 \varepsilon_1,$$

with the aid of Lemma 2.1 and (2.27). Now, (2.45) follows from this and the fact that

$$\lim_{\mu \to -\infty} B_{n,\tau,\mu} = B_{n,\tau,*},\ n \in \mathbb{N},$$

by applying Lebesgue's dominated convergence theorem.

2.4.3. *Step 3: End of the proof.* Putting (2.41) and (2.44)-(2.45) together, we obtain that

$$S_\tau = \sum_{n=1}^{+\infty} \left(A_{n,\tau,*} + B_{n,\tau,*} \right). \tag{2.54}$$

Bearing in mind that $\mathrm{Im} \left(\left| \lambda_\tau^+ - \lambda_{j,n} \right| \right) = 2\tau$ for $j = 1,2$ and all $n \in \mathbb{N}$, we infer from (2.21) with $(X,p) = (\partial\Omega, 2)$, (2.38), and (2.44)-(2.45) that

$$|A_{n,\tau,*}| \leqslant \frac{|\lambda_{1,n} - \lambda_{2,n}| \, \|\psi_{1,n}\|_{L^2(\partial\Omega)} \|\psi_{2,n}\|_{L^2(\partial\Omega)}}{\tau^2} c_*^2$$

and

$$|B_{n,\tau,*}| \leqslant \frac{\left(\|\psi_{1,n}\|_{L^2(\partial\Omega)} + \|\psi_{2,n}\|_{L^2(\partial\Omega)} \right) \|\psi_{1,n} - \psi_{2,n}\|_{L^2(\partial\Omega)}}{\tau} c_*^2.$$

Therefore, we have

$$\lim_{\tau \to +\infty} A_{n,\tau,*} = \lim_{\tau \to +\infty} B_{n,\tau,*} = 0,\ n \in \mathbb{N}.$$

and it follows from (2.54) for any arbitrary fixed $N \in \mathbb{N}$, that

$$\limsup_{\tau \to +\infty} |S_\tau| \leqslant \limsup_{\tau \to +\infty} \sum_{n=N}^{+\infty} |A_{n,\tau,*}| + \limsup_{\tau \to +\infty} \sum_{n=N}^{+\infty} |B_{n,\tau,*}|. \tag{2.55}$$

Moreover, setting $\delta_N := \sup_{n \geqslant N} |\lambda_{1,n} - \lambda_{2,n}|$, we infer from (2.38) and (2.44) that

$$\begin{aligned} \sum_{n=N}^{+\infty} |A_{n,\tau,*}| &\leqslant \delta_N \sum_{n=N}^{+\infty} \left| \frac{\langle f_\tau^-, \psi_{1,n} \rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+ - \lambda_{1,n}} \right| \left| \frac{\langle f_\tau^+, \psi_{2,n} \rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+ - \lambda_{2,n}} \right| \\ &\leqslant \delta_N \left(\sum_{n=N}^{+\infty} \left| \frac{\langle f_\tau^-, \psi_{1,n} \rangle_{L^2(\partial\Omega)}}{\lambda_\tau^- - \lambda_{1,n}} \right|^2 \right)^{1/2} \left(\sum_{n=N}^{+\infty} \left| \frac{\langle f_\tau^+, \psi_{2,n} \rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+ - \lambda_{2,n}} \right|^2 \right)^{1/2}. \end{aligned}$$

In the last line we used the Cauchy-Schwarz inequality and the identity $|\lambda_\tau^- - \lambda_{1,n}| = |\lambda_\tau^+ - \lambda_{1,n}|$. Therefore, applying Lemma 2.1 and (2.27), we obtain that

$$\sum_{n=N}^{+\infty} |A_{n,\tau,*}| \leqslant \delta_N \left\| u_{1,\lambda_\tau^-}^- \right\|_{L^2(\partial\Omega)} \left\| u_{2,\lambda_\tau^+}^+ \right\|_{L^2(\partial\Omega)} \leqslant \frac{(M+2)^2}{4} c_*^2 \delta_N,$$

which entails

$$\limsup_{\tau \to +\infty} \sum_{n=N}^{+\infty} |A_{n,\tau,*}| \leqslant \frac{(M+2)^2}{4} c_*^2 \left(\sup_{n \geqslant N} |\lambda_{1,n} - \lambda_{2,n}| \right). \tag{2.56}$$

Similarly, using (2.38) and (2.45), we can upper bound $\sum_{n=N}^{+\infty}|B_{n,\tau,*}|$ by

$$\begin{aligned}&\sum_{n=N}^{+\infty}\|\psi_{1,n}-\psi_{2,n}\|_{L^2(\partial\Omega)}\left(\left\|f_\tau^+\right\|_{L^2(\partial\Omega)}\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|+\|f_\tau^-\|_{L^2(\partial\Omega)}\left|\frac{\langle f_\tau^+,\psi_{2,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|\right)\\ \leqslant\ &\varepsilon_N\left(\left\|f_\tau^+\right\|_{L^2(\partial\Omega)}\left(\sum_{n=N}^{+\infty}\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|^2\right)^{1/2}+\|f_\tau^-\|_{L^2(\partial\Omega)}\left(\sum_{n=N}^{+\infty}\left|\frac{\langle f_\tau^+,\psi_{2,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|^2\right)^{1/2}\right),\end{aligned}$$

where $\varepsilon_N:=\left(\sum_{n=N}^{+\infty}\|\psi_{1,n}-\psi_{2,n}\|_{L^2(\partial\Omega)}^2\right)^{1/2}$. Next, applying Lemma 2.1 we get for all $N\in\mathbb{N}$ that $\sum_{n=N}^{+\infty}\left|\frac{\langle f_\tau^+,\psi_{2,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|^2\leqslant\left\|u_{2,\lambda_\tau^+}^+\right\|_{L^2(\Omega)}^2$, and we have

$$\sum_{n=N}^{+\infty}\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{2,n}}\right|^2\leqslant 4\sum_{n=N}^{+\infty}\left|\frac{\langle f_\tau^-,\psi_{1,n}\rangle_{L^2(\partial\Omega)}}{\lambda_\tau^+-\lambda_{1,n}}\right|^2\leqslant 4\left\|u_{1,\lambda_\tau^-}^-\right\|_{L^2(\Omega)}^2,$$

for every $N\geqslant N_0$, by virtue of (2.53). Therefore, in light of (2.21) with $(X,p)=(\partial\Omega,2)$ and (2.27), we have

$$\sum_{n=N}^{+\infty}|B_{n,\tau,*}|\leqslant\varepsilon_N\left(2\left\|f_\tau^+\right\|_{L^2(\partial\Omega)}\left\|u_{1,\lambda_\tau^-}^-\right\|_{L^2(\Omega)}+\|f_\tau^-\|_{L^2(\partial\Omega)}\left\|u_{2,\lambda_\tau^+}^+\right\|_{L^2(\Omega)}\right)\leqslant\frac{3(M+2)}{2}c_*^2\varepsilon_N,$$

provided $N\geqslant N_0$, and hence

$$\limsup_{\tau\to+\infty}\sum_{n=N}^{+\infty}\left|B_{n,\lambda_\tau^+,*}\right|\leqslant\frac{3(M+2)}{2}c_*^2\left(\sum_{n=N}^{+\infty}\|\psi_{1,n}-\psi_{2,n}\|_{L^2(\partial\Omega)}^2\right)^{1/2},\ N\geqslant N_0.$$

Putting this together with (2.55)-(2.56), we obtain

$$\limsup_{\tau\to+\infty}|S_\tau|\leqslant c\left(\left(\sum_{n=N}^{+\infty}\|\psi_{1,n}-\psi_{2,n}\|_{L^2(\partial\Omega)}^2\right)^{1/2}+\sup_{n\geqslant N}|\lambda_{1,n}-\lambda_{2,n}|\right),\ N\geqslant N_0,\tag{2.57}$$

the constant $c:=(M+2)(M+8)c_*^2/4$ being independent of N. Now, by sending N to $+\infty$ in the right hand side of the above estimate, we get that $\limsup_{\tau\to+\infty}|S_\tau|=0$. Thus, we have $\lim_{\tau\to+\infty}S_\tau=0$, by virtue of Proposition 2.4, which entails in the same way as in Section 2.3, that $q_1=q_2$ in Ω. This terminates the proof of Theorem 1.5

2.5. **The stability issue.** In this section we establish at the expense of stronger regularity on the potential that, provided it is known on the boundary $\partial\Omega$, it can be Hölder-stably determined by its BSD.

2.5.1. *Notations and stability inequality.* We stick with the notations of Section 2. In particular, given two real-valued potentials q_j for $j=1,2$, we denote by $\{\lambda_{j,n},\ n\in\mathbb{N}\}$ the sequence of the eigenvalues of A_{q_j} arranged in non-decreasing order (and repeated with the multiplicity) and for all $n\in\mathbb{N}$, we write $\psi_{j,n}$ instead of $\partial_\nu\varphi_{j,n}$, where $\{\varphi_{j,n},\ n\in\mathbb{N}\}$ is a $L^2(\Omega)$-orthonormal basis of eigenvectors of A_{q_j}, such that $A_{q_j}\varphi_{j,n}=\lambda_{j,n}\varphi_{j,n}$.

Theorem 2.5. *For $M\in(0,+\infty)$ fixed, pick q_1 and q_2 in $L^\infty(\Omega,\mathbb{R})\cap H^1(\Omega)$, such that*

$$\|q_j\|_{L^\infty(\Omega)}+\|q_j\|_{H^1(\Omega)}\leqslant M,\ j=1,2,\tag{2.58}$$

and

$$q_1 = q_2 \text{ on } \partial\Omega. \tag{2.59}$$

Assume moreover that

$$\sum_{n=1}^{+\infty} \|\psi_{1,n} - \psi_{2,n}\|^2_{L^2(\partial\Omega)} < \infty. \tag{2.60}$$

Then, we have

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C \limsup_{n\to+\infty} |\lambda_{1,n} - \lambda_{2,n}|^{2/(d+2)},$$

for some positive constant C, that depends only on Ω and M.

2.5.2. *Proof of Theorem 2.5.* Let us recall from (2.57) that $N \geqslant N_0$, the estimate

$$\limsup_{\tau\to+\infty} |S_\tau| \leqslant c \left(\left(\sum_{n=N}^{+\infty} \|\psi_{1,n} - \psi_{2,n}\|^2_{L^2(\partial\Omega)} \right)^{1/2} + \sup_{n\geqslant N} |\lambda_{1,n} - \lambda_{2,n}| \right)$$

holds for some positive constant c that is independent of N and ξ. Thus, in light of (2.60), we get upon sending N to infinity, that

$$\limsup_{\tau\to+\infty} |S_\tau| \leqslant c \limsup_{n\to\infty} |\lambda_{1,n} - \lambda_{2,n}|. \tag{2.61}$$

Further, we recall from Proposition 2.4 that

$$\lim_{\tau\to+\infty} S_\tau = \int_\Omega q(x) e^{-x\cdot\xi} dx = (2\pi)^{d/2} \widehat{q}(\xi),$$

where q is the same as in (2.34) and $\widehat{q}$ stands for the Fourier transform $\mathcal{F}q$ of q, defined by (2.33). This, (2.61) and the basic estimate $|\lim_{\tau\to+\infty} S_\tau| \leqslant \limsup_{\tau\to+\infty} |S_\tau|$, yield $|\widehat{q}(\xi)| \leqslant (2\pi)^{-d/2} c \limsup_{n\to+\infty} |\lambda_{1,n} - \lambda_{2,n}|$, uniformly in $\xi \in \mathbb{R}^d$. Thus, we have

$$\|\widehat{q}\|_{L^\infty(\mathbb{R}^d)} \leqslant c \limsup_{n\to+\infty} |\lambda_{1,n} - \lambda_{2,n}|. \tag{2.62}$$

upon substituting $(2\pi)^{-d/2}c$ for c.

On the other hand, we infer from (2.34) and from the Plancherel theorem that

$$\|q_1 - q_2\|^2_{L^2(\Omega)} = \|q\|^2_{L^2(\mathbb{R}^d)} = \|\widehat{q}\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\widehat{q}(\xi)|^2 d\xi. \tag{2.63}$$

Let $R \in (1, +\infty)$ be fixed. Set $B_R := \{\xi \in \mathbb{R}^d,\ |\xi| \leqslant R\}$ and notice from (2.63) that

$$\|q_1 - q_2\|^2_{L^2(\Omega)} = \int_{B_R} |\widehat{q}(\xi)|^2 d\xi + \int_{\mathbb{R}^d\setminus B_R} |\widehat{q}(\xi)|^2 d\xi. \tag{2.64}$$

The first term in the right hand side of (2.64) is easily treated, as we have

$$\int_{B_R} |\widehat{q}(\xi)|^2 d\xi \leqslant \tilde{c} R^d \|\widehat{q}\|^2_{L^\infty(B_R)}, \tag{2.65}$$

where $\tilde{c}$ is a positive constant that is independent of R. Further, since $q_1 - q_2 \in H^1_0(\Omega)$ by (2.59), we see that $q \in H^1(\mathbb{R}^d)$, so we get

$$\int_{\mathbb{R}^d\setminus B_R} (1 + |\xi|^2) |\widehat{q}(\xi)|^2 d\xi = \|q\|^2_{H^1(\mathbb{R}^d)} = \|q_1 - q_2\|^2_{H^1(\Omega)} \leqslant \left(\|q_1\|_{H^1(\Omega)} + \|q_2\|_{H^1(\Omega)} \right)^2 \leqslant 4M^2,$$

from (2.58), and consequently

$$\int_{\mathbb{R}^d \setminus B_R} |\widehat{q}(\xi)|^2 \, d\xi \leqslant R^{-2} \int_{\mathbb{R}^d \setminus B_R} (1+|\xi|^2) \, |\widehat{q}(\xi)|^2 \, d\xi \leqslant 4M^2 R^{-2}.$$

Putting this and (2.64)-(2.65) together, we find that

$$\|q_1 - q_2\|_{L^2(\Omega)}^2 \leqslant \tilde{c} \left(R^d \, \|\widehat{q}\|_{L^\infty(B_R)}^2 + R^{-2} \right), \tag{2.66}$$

upon possibly substituting $\max(\tilde{c}, 4M^2)$ for $\tilde{c}$.

Set $\delta := \limsup_{n \to +\infty} |\lambda_{1,n} - \lambda_{2,n}|$. We shall examine the two cases $\delta \in (0,1)$ and $\delta \in [1, +\infty)$ separately. In the first case we plug the estimate $\|\widehat{q}\|_{L^\infty(B_R)} \leqslant c\delta$, arising from (2.62), in (2.66), choose $R = \delta^{-2/(d+2)}$, and get

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C \delta^{2/(d+2)}, \tag{2.67}$$

with $C := \left(\tilde{c}(1+c^2) \right)^{1/2}$. In the second case, where $\delta \geqslant 1$, we have obviously

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant \|q_1\|_{L^2(\Omega)} + \|q_2\|_{L^2(\Omega)} \leqslant 2M \leqslant 2M \delta^{2/(d+2)},$$

so the desired result follows from this and (2.67).

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