Permutating quantum eigenmodes by using the quasi-adiabatic motion of a wall

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in collaboration with R. Joly and D. Turaev

Identification and Control: some challenges
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Let the Schrödinger equation in $\mathscr{H} = L^2((0,1),\mathbb{C})$

$$\begin{cases} i\partial_t u(t,x) = -\partial_{xx}^2 u(t,x) + V(x,t)u(t,x), & x \in (0,1), \\ u(t,0) = u(t,1) = 0, & t \in [0,T]. \end{cases} \tag{SE}$$

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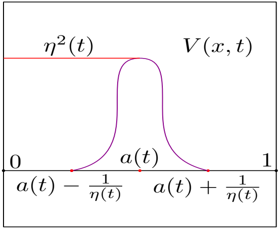
$$a \in C^2((0,T),(0,1)), \qquad \eta \in C^2((0,T),\mathbb{R}^+),$$

$$\rho \in C^2(\mathbb{R},\mathbb{R}^+), \quad supp(\rho) \subset [-1,1], \quad \int_{\mathbb{R}} \rho(s)ds = 1.$$

Aim: We are interested in controlling the eigenmodes of the Schrödinger equation by a suitable motion of

$$V(t,x) := \eta^2(t)\rho(\eta(t)(x-a(t))).$$



In $L^2((0,a)\cup(a,1),\mathbb{C})$ with $a\in(0,1)$, we consider the Hamiltonian

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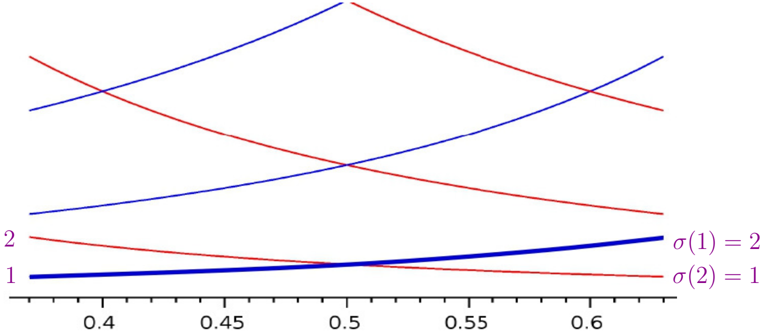
By moving $a \in [\sqrt{2}/4, 1-\sqrt{2}/4]$, we can steer the first eigenmode to the second (and vice versa) through a permutation σ.



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$$\forall \psi \in D(H^{a,\alpha}), \quad \alpha\psi(a) + (1-\alpha)(\partial_x u(a^+) - \partial_x u(a^-)) = 0,$$

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- He moves a from $1 - \sqrt{2}/4$ to $\sqrt{2}/4$ and the first eigenmode becomes the second one (as in the previous slide).
- He tracks the path of the eigenmode while the parameter α goes from 1 to 0.

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- Although, the adiabatic motion preserves the ordering of the eigenmodes as the spectrum of $-\partial^2_{xx} + V$ is always simple.

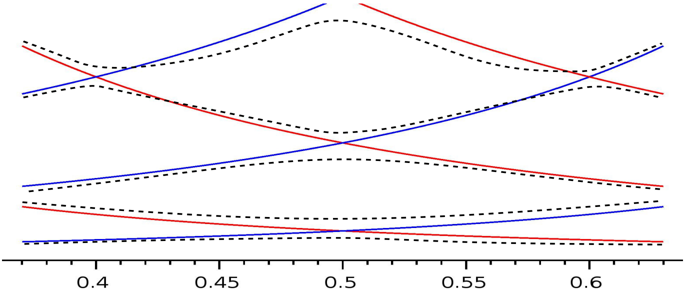

Figure: *The dashed curves represent the eigenvalues of* $-\partial^2_{xx} + V$.

The permutation happens when the eigenvalues of $-\partial_{xx}^2 + V(t,\cdot)$ with $t \in [0, T]$ follow the ones of $H^{a(t)} = -\partial_{xx}^2$ with domain

$$D(H^{a(t)}) = H^2 \cap H_0^1((0, a(t)) \cup (a(t), 1), \mathbb{C}).$$

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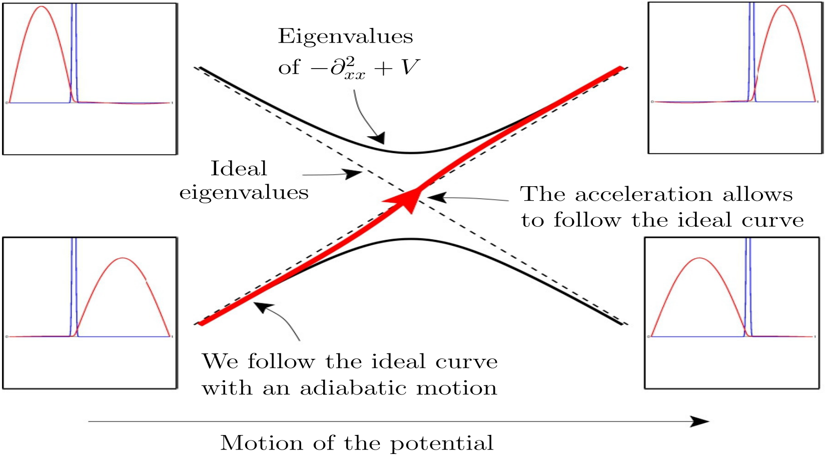
$\implies$ We need to accelerate the dynamics in proximity of the crossing points of the ideal dynamics.



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- Due to the very high amplitude of the potential, the map

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- Nevertheless, we ensure that such perturburbations are negligible when we consider functions which almost vanish in the support of V and the accelerations last short time.

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Let Γ_0^T the propagator associated to (SE) in the time $[0, T]$.

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Theorem (D., Joly, Turaev)

Let $\varepsilon > 0$. There exist $T > 0$,

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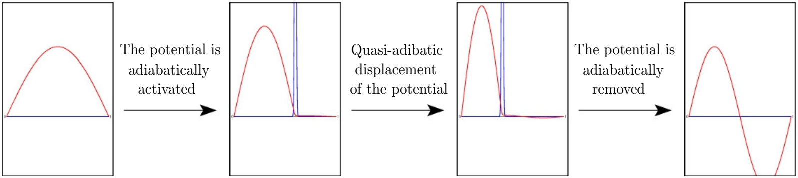
such that there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ with $|\alpha_1| = |\alpha_2| = 1$ such that

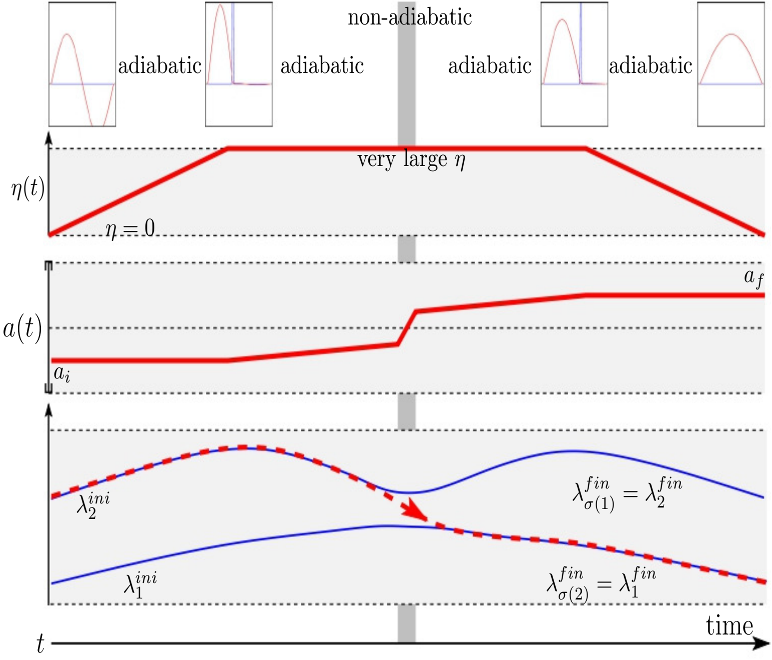
$$\| \; \Gamma_0^T \sin(\pi x) \; - \; \alpha_1 \sin(2\pi x) \; \|_{L^2} \leq \varepsilon \; ,$$

$$\| \; \Gamma_0^T \sin(2\pi x) \; - \; \alpha_2 \sin(\pi x) \; \|_{L^2} \leq \varepsilon \; .$$

Construction of the motion

- **The first "vertical" adiabatic motion.** We increase η up to obtain a large and thin wall located at $1-\sqrt{2}/4$. This dynamics is adiabatic in order to keep track of the mode.
- **"Horizontal" quasi-adiabatic motion.** We move the wall from the location $1-\sqrt{2}/4$ to $\sqrt{2}/4$.
- **Last "vertical" adiabatic motion.** We adiabatically decrease the potential up to extinction.




non-adiabatic
adiabatic
adiabatic
adiabatic
adiabatic
very large η
$\eta(t)$
$\eta = 0$
$a(t)$
a_i
a_f
λ_2^{ini}
λ_1^{ini}
$\lambda_{\sigma(1)}^{fin} = \lambda_2^{fin}$
$\lambda_{\sigma(2)}^{fin} = \lambda_1^{fin}$
t
time

Further results

Let $\{\eta_j\}_{j\leq J} \subset C^\infty([0,T],\mathbb{R}^+)$, $\{a_j\}_{j\leq J} \subset C^\infty([0,T],(0,1))$ and

$$V(t,x) \;=\; \sum_{j=1}^{J} \eta_j(t)\rho(\eta_j(t)(x-a_j(t))). \qquad (1)$$

Proposition (D., Joly, Turaev)

Let $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ be any permutation. For all $N \in \mathbb{N}^*$ and $\varepsilon > 0$, there exist $J \in \mathbb{N}^*$, $T > 0$, $\{\eta_j\}_{j\leq J} \subset C^\infty([0,T],\mathbb{R}^+)$ and $\{a_j\}_{j\leq J} \in C^\infty([0,T],(0,1))$ such that the following property holds. Let Γ_0^T the propagator generated by the (SE) in the time interval $[0,T]$ with the potential V defined by (1). For all $k \leq N$, there exists $\alpha_k \in \mathbb{C}$ with $|\alpha_k| = 1$ such that

$$\left\| \, \Gamma_0^T \sin(k\pi x) \; - \; \alpha_k \sin(\sigma(k)\pi x) \, \right\|_{L^2} \leq \varepsilon \; .$$

Thank you for your attention!