On the optimal control of systems with missing data

Identification and Control : some challenges

Abdelhak HAFDALLAH

UNIVERSITY LARBI TEBESSI, TEBESSA, ALGERIA
Department of Mathematics and Computer Science
Faculty of Exact Sciences and Sciences of Nature and Life

Monastir, June 19th, 2019

Overview

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- Generalities and basic concepts on optimal control of distributed systems with incomplete data

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- Optimal control of electromagnetic wave displacement with an unknown velocity of propagation and a missing boundary condition

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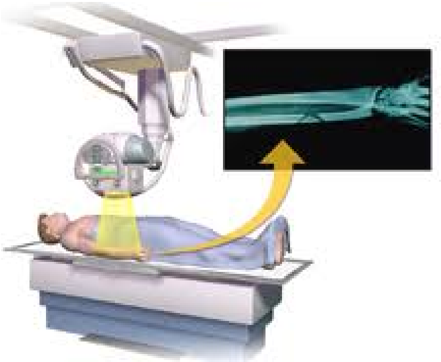
X-rays penetration in a human body



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(Lions, 1992)We say that $u \in \mathcal{U}_{ad}$ is a no-regret control for (5) (6) if u solves

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Remark

Of course, the problem (10) *is defined only for controls such that*

$$\sup_{g \in G} \left(J(v,g) - J(0,g) \right) < \infty \tag{11}$$

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$$K = \left\{ v \in \mathcal{U}_{ad} : \langle S(v), g\rangle_{G',G} = 0 \ \forall g \in G \right\}$$

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By (12) and Legendre transform we get

$$\sup_{g \in G} \left(2 \langle S(v), g \rangle_{G',G} - \gamma \|g\|_G^2 \right) = \frac{1}{\gamma} \|S(v)\|_{G'}^2$$

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The sequence of low-regret controls converges weakly in $\mathcal{U}_{ad}$ when $\gamma \to 0$ to the unique no-regret control u solution to (10).

Proposition

The low-regret control u_γ, solution to (5) (14) (15) is characterized by the following optimality system

$$\left\{ \begin{array}{c} \mathcal{A} y_\gamma = f + \mathcal{B} u_\gamma, \\ \mathcal{A}^* \zeta_\gamma = C^* C \left(y_\gamma - y\left(0,0\right)\right), \\ \mathcal{A} \rho_\gamma = \frac{1}{\gamma} \beta \beta^* \zeta_\gamma, \\ \mathcal{A}^* p_\gamma = C^* \left(C y_\gamma - z_d \right) + C^* C \rho_\gamma, \\ \left(\mathcal{B}^* p_\gamma + N u_\gamma, v - u_\gamma \right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \end{array} \right. \tag{16}$$

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A motivating example, in biomedical phenomena the X-rays could damage cells, to avoid their harmful effects we have to make the displacement and consequently the energy suitable for the burden of living cells.

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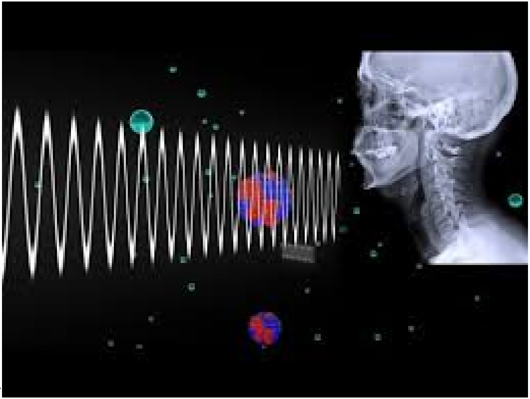

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We want to get an optimal control independent of incomplete data.
To achieve this goal, we shall use the concepts of averaged control and no-regret control introduced for the first and second kind of incomplete data respectively, to introduce the *averaged no-regret control*.

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$$J(v,g) = \left\| \int_{\sigma_1}^{\sigma_2} y(v,g,\sigma)\, d\sigma - z_d \right\|_{L^2(Q)}^2 + N \|v\|_{L^2(Q)}^2 \qquad (51)$$

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Theorem

There exists a unique averaged low-regret control u_γ solution to (50) (55) (56).

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$$\begin{cases} \frac{d^2 y_\gamma}{dt^2} - \sigma^2 \Delta y_\gamma = u_\gamma, \\ \frac{d^2 \xi_\gamma}{dt^2} - \Delta \xi_\gamma = \frac{1}{t} \int_{\sigma_1}^{\sigma_2} y\left(u_\gamma, 0; \sigma\right) d\sigma, \\ \frac{d^2 \rho_\gamma}{dt^2} - \Delta \rho_\gamma = 0, \\ \frac{d^2 p_\gamma}{dt^2} - \sigma^2 \Delta p_\gamma = \int_{\sigma_1}^{\sigma_2} y\left(u_\gamma, 0; \sigma\right) d\sigma - z_d + \left(\sigma_2 - \sigma_1\right) \frac{\rho_\gamma}{t} & in\ Q, \\ y_\gamma = 0, \xi_\gamma = 0,\ \rho_\gamma = -\frac{t^2}{\gamma} \frac{\partial \xi_\gamma}{\partial \nu}\ ,\ p_\gamma = 0 & on\ \Sigma, \\ y_\gamma\left(x, 0\right) = 0, \frac{dy_\gamma}{dt}\left(x, 0\right) = 0, \\ \xi_\gamma\left(x, T\right) = 0, \frac{d\xi_\gamma}{dt}\left(x, T\right) = 0, \\ \rho_\gamma\left(x, 0\right) = 0, \frac{d\rho_\gamma}{dt}\left(x, 0\right) = 0, \\ p_\gamma\left(x, T\right) = 0, \frac{dp_\gamma}{dt}\left(x, T\right) = 0 & in\ \Omega, \end{cases} \tag{57}$$

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with the variational inequality:

$$\left(\int^{\sigma_2} p_\gamma\left(\sigma\right) d\sigma + N u_\gamma, v - u_\gamma \right) \geq 0 \qquad \text{for every } v \in \mathcal{U}_{ad}.$$

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THANK YOU FOR YOUR ATTENTION