Theoretical and numerical aspects related to the energy of some Timoshenko systems

Makram Hamouda
Faculty of Sciences of Tunis

Identification and Control, Monastir
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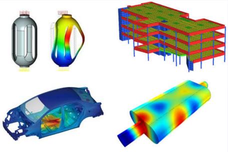
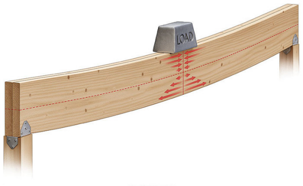
Outline

Vibration is everywhere ! !

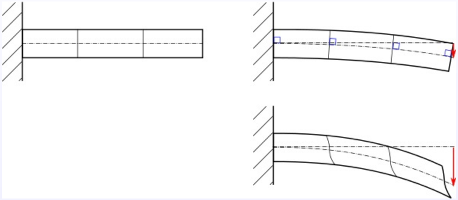


Mechanical structures such as beams and plates are a central part of life today, their vibration properties are extensively investigated by many researchers. These vibrations are undesirable because of their damaging and destructing nature.



To reduce these harmful vibrations, several control mechanisms have been designed. In order to do that, it is natural to model and understand the corresponding equations of these problems.

The 1D Timoshenko model (1921)



$$\begin{cases} \rho\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & in \;\; (0, L) \times (0, +\infty)\,, \\ I_\rho\psi_{tt} - (EI\psi_x)_x + K(\varphi_x + \psi) = 0, & in \;\; (0, L) \times (0, +\infty)\,, \end{cases}$$

t : the time variable.

x : the space variable.

φ : the displacement vector.

ψ : the rotation angle of the filament.

ρ, E, I, K are positive constants.

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0,1)\times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + d\psi_t = 0, & \text{in } (0,1)\times \mathbb{R}_+. \end{cases} \tag{1}$$

"$d\psi_t$" $\rightsquigarrow$ several types of dissipation have been studied.
The common point in all these works is the following condition :

“Equal wave speeds” Condition

$$\frac{\mathbf{k}}{\rho_1} = \frac{\mathbf{b}}{\rho_2}.$$

The system (1) is exponentially stable if and only if the wave speeds are equal.

Mũnoz Rivera, J.E., Racke, R. : Global stability for damped Timoshenko systems. Disc. Cont. Dyn. Sys. 9 (2003), 1625–1639.

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + g(\psi_t) = 0, & \text{in } (0,1)\times\mathbb{R}_+. \end{cases} \quad (2)$$

$$E(t) := \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k(\varphi_x+\psi)^2\right) dx.$$

Upper energy estimate : $(\frac{k}{\rho_1} = \frac{b}{\rho_2})$ $E(t) \leq C(E(0))(H')^{-1}\left(\frac{1}{t}\right)$.

Lower energy estimate : $E(t) \geq C(E_1(0))\left((H')^{-1}\left(\frac{1}{t-T_0}\right)\right)^2$.

Alabau-Boussouira, F. : Asymptotic behavior for Timoshenko beams subject to a single nonlinear feedback control. Nonlinear Differ. Equ. Appl. 14 (5–6), 643–669 (2007)

Dafermos, C. : Asymptotic behavior of solutions of evolution equations. In : Nonlinear Evolution Equations . Center Univ. Wisconsin, vol. 40, pp. 103–123. Academic Press, New York (1978)

Optimality of Timoshenko system with thermoelasticity

Analytical Results

Joint work with

Abdelaziz Soufyane, Ahmed BCHATNIA and Sabrine Chebbi.

LOWER BOUND AND OPTIMALITY FOR A NONLINEARLY DAMPED TIMOSHENKO SYSTEM WITH THERMOELASTICITY

Timoshenko system with thermoelasticity

Fernández Sare and **Racke** (2009) introduced the heat conduction through Cattaneo's law

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0,1) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x = 0, & \text{in } (0,1) \times \mathbb{R}_+, \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0, & \text{in } (0,1) \times \mathbb{R}_+, \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,1) \times \mathbb{R}_+. \end{cases}$$

$\rightarrow$ **No exponential** decay even though the wave speeds are equal.
Goal : Obtain the lower bound and investigate the optimality of the following system, in $(0,1) \times \mathbb{R}_+$,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x + a(x)g(\psi_t) = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0. \end{cases} \qquad (3)$$

Initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & \forall\ x \in\ (0,1), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), & \forall\ x \in\ (0,1), \\ \theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x), & \forall\ x \in\ (0,1), \end{cases}$$

Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0.$$

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times (L^2(0,1))^3.$$

The energy for $U = (\varphi, \psi, \theta, q)$

$$E(U)(t) := \frac{1}{2}\int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3 \theta^2 + \tau q^2)dx.$$

Assumptions :

- (H_0) : a is a **smooth function**, $a(x) \geq 0$, $x \in (0,1)$, $a > 0$ in a nonempty subset $(0,1)$ of Γ.

$$(H_1) \begin{cases} g : \mathbb{R} \to \mathbb{R} \text{ is a nondecreasing } \mathcal{C}^0 \text{ function} \\ \text{such that for every } \epsilon \in (0,1), \exists c_1 > 0, c_2 > 0, \\ \text{an increasing odd function } g_0 \in C^1(0,+\infty), \\ g_0(0) = 0 \text{ such that} \\ \begin{cases} g_0(|(s)|) \leq |g(s)| \leq g_0^{-1}(|(s)|), & \text{for all } |s| \leq \epsilon, \\ c_1|s| \leq |g(s)| \leq c_2|s|, & \text{for all } |s| \geq \epsilon. \end{cases} \end{cases}$$

- We introduce

$$H(x) = \sqrt{x}g(\sqrt{x}), \quad \Lambda(x) = \frac{H(x)}{xH'(x)}$$

H is a strictly convex $\mathcal{C}^1-$function from $[0, r_0^2]$ onto $\mathbb{R}$.

Dissipation : $E'(t) = -\beta \int_0^1 q^2 dx - \int_0^1 a(x)\psi_t g(\psi_t) dx \leq 0.$

$$(HC)\left\{\begin{array}{l}\text{Let } (\varphi,\psi) \text{ be a weak solution of } (2)\\ \text{if } \psi_t \equiv 0 \text{ on } \Gamma \text{ then } (\varphi,\psi) \equiv (0,0).\end{array}\right.$$

$$(H_2)\left\{\begin{array}{l}\exists r_0 > 0 \text{ such that the function } H : [0, r_0^2] \mapsto \mathbb{R}\\ \text{is strictly convex on } [0, r_0^2],\\ \text{and either } 0 < \liminf_{x\to 0} \Lambda(x) \leq \limsup_{x\to 0} \Lambda(x) < 1,\\ \text{or there exists } \mu > 0 \text{ such that}\\ 0 < \liminf_{x\to 0}\left(\frac{\Psi(\mu x)}{\mu x}\int_x^{z_1}\frac{1}{H(y)}dy\right), \text{ and } \limsup_{x\to 0}\Lambda(x) < 1,\\ \text{for some } z_1 \in (0, z_0] \text{ and for all } z_0 > 0.\end{array}\right.$$

Challenges

- Is our system (3) strongly stable ?
- If we obtain a different equilibrium state ($E(t) \to$ constant $\neq 0$ as $t \to \infty$), how can we characterize the decay rate of the energy ?
- Can we obtain lower estimates for the new equilibrium state ?

Results

- Strong stability theorem
  Stability number : $\chi = \left(\tau_0 - \frac{\rho_1}{k\rho_3}\right)\left(\rho_2 - \frac{\rho_1 b}{k}\right) - \frac{\rho_1 \beta^2 \rho_1}{k\rho_3}$.
- Lower energy estimates.
- Optimality for an explicit example.

Theorem 1 (New equilibrium state)

Assume that (H_0), (H_1) and (HC) hold. Then,

- for all $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in \mathcal{H}$, the energy E satisfies

$$\lim_{t\to\infty} E(t,U) = E_\infty, \qquad \forall\ U \in \mathcal{C}([0,+\infty),\mathcal{H})$$

  where E_∞ is the energy of $Z \in \omega(U_0)$.
- The energy $\mathcal{E}(t)$ satisfies

$$\mathcal{E}(t) \to 0, \quad \text{as} \quad t \to \infty.$$

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 \rho_1 {\varphi_t}^2 + \rho_2 {\psi_t}^2 + b{\psi_x}^2 + k(\varphi_x+\psi)^2 + \rho_3(\theta-\theta_0(0))^2 + \tau q^2 dx.$$

Sketch of the proof

- $E_\star(t)$ is a non-increasing function (the first-order energy)

$$E_\star(t) := \frac{1}{2}\int_0^1 (\rho_1\varphi_{tt}^2+\rho_2\psi_{tt}^2+b\psi_{tx}^2+k(\varphi_{tx}+\psi_t)^2+\rho_3\theta_t^2+\tau q_t^2)dx.$$

- For the initial data $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in D(A)$, the orbit of U_0 given by $\gamma(U_0) = \cup_{t\geq 0}\mathcal{T}(t)U_0$ is relatively compact in $\mathcal{H}$.
- Thanks to the Dafermos strong stability

$$\lim_{t\to\infty} E(t, U) = E_\infty, \qquad \forall\ U \in \mathcal{C}([0, +\infty), \mathcal{H}).$$

Theorem 2 (Lower bound)

Assume that (H_0), (H_1) and (H_2) hold.
For all smooth initial data U_0 $(\neq 0)$, $\exists$ $T_0 > 0$ and $T_1 > 0$, such that the energy $\mathcal{E}$ satisfies the lower estimate

$$\frac{1}{\gamma^2 C_\sigma^2}\left(H'^{-1}\left(\frac{1}{t-T_0}\right)\right)^2 \leq \mathcal{E}(t), \qquad \forall\ t \geq T_1 + T_0, \tag{4}$$

where σ is a positive constant $\sigma = \frac{\alpha_a}{\rho_2} + \frac{\beta r_0}{\tau C_1}$,

$$\alpha_a = \|a\|_{L^\infty(0,1)},$$

and

$$\gamma = \frac{4}{\rho_2}\sqrt{E_\star(0)}.$$

- Example 1. Let $g(s) = s^p$, $\forall s \in (0, r_0^2]$ for all $p > 1$.

$$\mathcal{E}(t) \geq c' \ t^{\frac{-4}{p-1}}.$$

- Example 2. Let $g(s) = \frac{1}{s} exp(-(ln(s))^2)$, for all $s \in (0, r_0^2]$,

$$\mathcal{E}(t) \geq c' \ exp(-4(ln(t)^{\frac{1}{2}})).$$

Both the upper and lower bounds yield the convergence to 0 as $t \to +\infty$, but with different rates.

- These results are not sufficient to prove optimality $\Rightarrow$ **one more step to optimality**.

Numerical Results

Joint work with

Sabrine Chebbi

Discrete Energy behavior of a damped Timoshenko system.

Numerical asymptotic behavior of the solution

- C. A. Raposo, J. A. D. Chuquipoma, J. A. J. Avila, M. L.Santos "Exponential Decay and Numerical Solution for a Timoshenko system with delay term in the internal feedback."(2013)
- D. S. Almeida Junior, "Conservative Semi-discrete Difference Schemes for Timoshenko Systems" (2014).
- M. L. Santos and Dilberto da S. Almeida Junior, "Exponential Decay to Dissipative Bresse System."(2010)

Challenges

- Our system is a **hyperbolic coupled** system $\rightarrow$ we start by the behavior of the wave equation (solution and energy).
- Design a discretization scheme based on a combination between **the finite elements** method and the **finite differences** one.
- Consider a Timoshenko system subject to different types of dissipation (undamped, linear damping and non-linear damping)

Results

- Energy Conservation Property of the Timoshenko Equations
- Discrete Energy behavior of a **damped** Timoshenko system.

1D Timoshenko system

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0. \end{cases} \tag{5}$$

The initial conditions

$$\begin{cases} \varphi(0,x) = \varphi_0(x), \psi(0,x) = \psi_0(x), & \forall x \in (0,1), \\ \varphi_t(0,x) = \varphi_1(x), \psi_t(0,x) = \psi_1(x), & \forall x \in (0,1), \end{cases}$$

The Dirichlet boundary conditions

$$\varphi = \psi = 0, \;\; x = 0, \;\; x = L.$$

The energy

$$E(U,t) := \frac{1}{2} \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 \right) dx.$$

We have $\frac{dE(t,U)}{dt} = 0 \Rightarrow$ The analytical energy conservation property

Semi-discrete scheme in finite element $u = \varphi_t,\ v = \psi_t$

We rewrite the system (5)

$$u_t = \varphi_{xx} + \psi_x,\ \ \forall x \in (0, L),\ \ t > 0,$$

$$v_t = \psi_{xx} - \varphi_x - \psi,\ \ \forall x \in (0, L),\ \ t > 0.$$

The variational setting

$$H_1(0,1) = \{\mathbf{u} \in \mathcal{C}(0,1) | \mathbf{u}(0) = \mathbf{u}(1) = 0\}.$$

Find u, v such that for all $t > 0$, $\forall\ \mathbf{u} \in H_1(0,1)$ and $\mathbf{v} \in H_1(0,1)$

$$(u_t, \mathbf{u}) + (v_t, \mathbf{v}) = (\psi_x, \mathbf{u}) - (\psi_x, \mathbf{v}_x) - (\varphi_x, \mathbf{v}) - (\psi, \mathbf{v}) - (\varphi_x, \mathbf{u}_x).$$

We consider the piecewise-linear function

$$w_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & x_{i-1} \le x \le x_i, \\ \dfrac{x_{i+1} - x}{h} & x_i \le x \le x_{i+1}, \\ 0 & x \in [0,1] \setminus [x_{i-1}, x_{i+1}], \end{cases}$$

The matrix formulation for the semi-discrete problem

$$\begin{cases} M\dfrac{dU}{dt} = -K\Psi + S\Psi, \\ \\ M\dfrac{dV}{dt} = -K\Psi - S\Phi - M\Psi, \\ \\ \dfrac{d\Phi}{dt} = U(t), \\ \\ \dfrac{d\Psi}{dt} = V(t), \end{cases} \tag{6}$$

where $U(t) = [u_h(x_0, t), \cdots, u_h(x_{Nx}, t)]^t$ and $V(t) = [v_h(x_0, t), \cdots, v_h(x_{Nx}, t)]^t$.
$M_{i,j} = (w_i, w_j)$ is the mass matrix, $K_{i,j} = (w_i', w_j')$ is the rigidity matrix and S is a matrix such that $S_{i,j} = (w_i', w_j)$.

Fully-discrete scheme in Finite Differences

The leapfrog time scheme

Find the discrete solutions $(\Phi^n, U^n, \Psi^n, V^n)$

$$\begin{cases} M\dfrac{U^{n+1}-U^{n-1}}{2\Delta t} = -K\Phi^n + S\Psi^n, \\ M\dfrac{V^{n+1}-V^{n-1}}{2\Delta t} = -K\Psi^n - S\Phi^n - M\Psi^n, \\ \dfrac{\Phi^{n+1}-\Phi^{n-1}}{2\Delta t} = U^n, \\ \dfrac{\Psi^{n+1}-\Psi^{n-1}}{2\Delta t} = V^n. \end{cases} \tag{7}$$

The initial data are evaluated using the explicit Euler approximation and we have

$$\begin{aligned} \Phi^0 &= \phi_0(x) \\ \Psi^0 &= \psi_0(x) \\ U^0 &= \phi_1(x) \\ V^0 &= \psi_1(x) \end{aligned} \tag{8}$$

$$\Phi^1 = \Phi^0 + \Delta t U^0 \tag{9}$$

$$\Psi^1 = \Psi^0 + \Delta t V^0 \tag{10}$$

$$MU^1 = MU^0 + \Delta t(-K\Phi^0 + S\Psi^0) \tag{11}$$

$$MV^1 = MV^0 + \Delta t(-K\Psi^0 + S^T\Phi^0 - M\Psi^0) \tag{12}$$

where S is an antisymmetric Matrix $S^T = -S$ (S^T is the adjoint matrix).

The discrete solutions $(\Phi^n, U^n, \Psi^n, V^n)$ are such that

$$\left\{\begin{array}{l} U^{n+1} = U^{n-1} - 2\Delta t \ \left(-M^{-1}K\Phi^n - M^{-1}S\Psi^n\right), \\ V^{n+1} = V^{n-1} - 2\Delta t \ \left(M^{-1}K\Psi^n - M^{-1}S\Phi^n - I\Psi^n\right), \\ \Phi^{n+1} = 2\Delta t \ U^n + \Phi^{n-1}, \\ \Psi^{n+1} = 2\Delta t \ V^n + \Psi^{n-1}. \end{array}\right. \tag{13}$$

$Nx = 50,\ L = 2,\ T = 10,\ h = 0.04\ ,\Delta t = c * \Delta x,\ c$

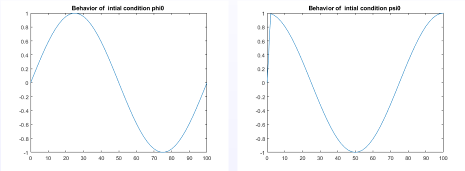


FIGURE – Behavior of the initial conditions φ_0 and ψ_0

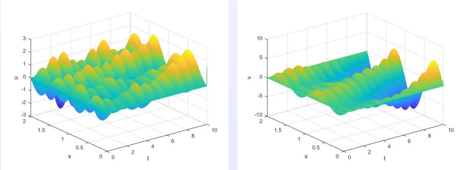


FIGURE – The numerical behavior of the solutions $u(x_i, t_n)$, $v(x_i, t_n)$.

Definition of the discrete energy

Let $\Delta t > 0$. We define the discrete energy

$$\begin{aligned} E^{n+\frac{1}{2}} \quad &:= \left(M\frac{\Phi^{n+2}-\Phi^{n}}{2\Delta t}, \frac{\Phi^{n+2}-\Phi^{n}}{2\Delta t} \right) \\ &+ \left(M\frac{\Psi^{n+2}-\Psi^{n}}{2\Delta t}, \frac{\Psi^{n+2}-\Psi^{n}}{2\Delta t} \right) \\ &+(K\Phi^{n}, \Phi^{n+2}) + ((K+M)\Psi^{n}, \Psi^{n+2}) \\ &+(S\Psi^{n}, \Phi^{n+2}) - (S\Phi^{n}, \Psi^{n+2}), \end{aligned} \tag{14}$$

(15)

Theorem 3 (Conservation of the discrete energy)

The discrete energy $E^{n+\frac{1}{2}}$ satisfies the following conservation property :

$$\frac{1}{2\Delta t}\left(E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}\right) = 0. \quad (16)$$

Hereafter we will denote by $\|.\|_M$ the norm $\|u\|_M^2 = (Mu, u)$.

M is a define positive matrix, $(M\Psi^n, V^{n+1}) \geq 0$

$\rightsquigarrow$ The numerical energy is positive $(E^{n+\frac{1}{2}} \geq 0)$ thanks to the CFL obtained in the following.

Theorem 4 (Conservation of the discrete energy)

Define the matrix $\mathcal{M}_1$ and $\mathcal{M}_2$ as

$$\mathcal{M}_1 = M - 2\Delta t^2(M + K) \tag{17}$$

and

$$\mathcal{M}_2 = (M - 2\Delta t^2 K) \tag{18}$$

Then, our discrete explicit scheme is stable if $E^{n+\frac{1}{2}} \geq 0$.
We infer the Courant-Friedrichs-Lewy (CFL) which reads as

$$\frac{\Delta t}{\Delta x} < \frac{1}{\sqrt{2}} \tag{19}$$

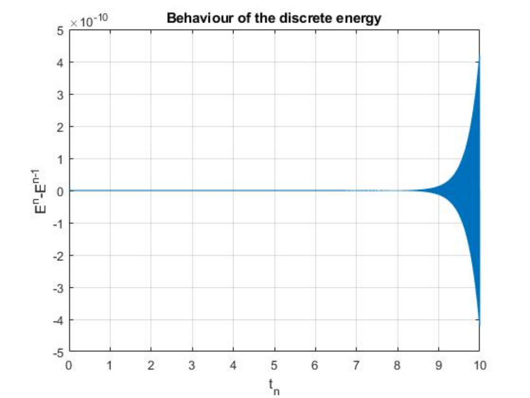


FIGURE – The undamped case : the conservative property of the discrete energy E^n defined by (14).

$$\begin{cases} \varphi_{tt} - (\varphi_x + \psi)_x = 0, & (x,t) \in (0,L) \times R_+, \\ \psi_{tt} - \psi_{xx} + (\varphi_x + \psi) + \mu\psi_t = 0, & (x,t) \in (0,L) \times R_+, \end{cases} \tag{20}$$

where, μ represents the damping coefficient.

Theorem 5 (The damped system)

For all $\Delta t > 0$, we have

$$\frac{1}{2\Delta t}\left(E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}\right) \leq 0. \tag{21}$$

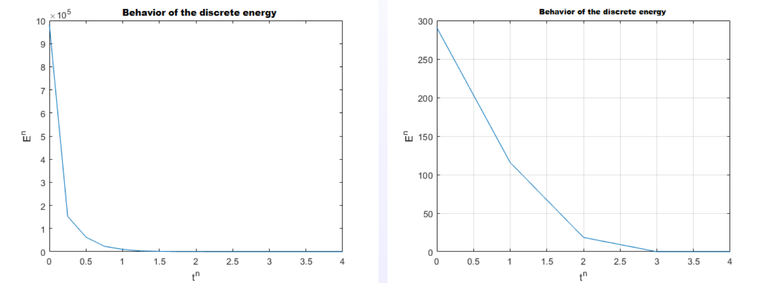


FIGURE – The damped case : the discrete energy expressed as a function of t_n (exponential decay with different initial data).

This figure represents $log(E^n)$ using $P1-$linear approximation.

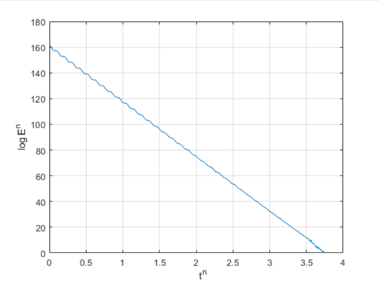


FIGURE – Exponential decay rate of discrete energy

The energy E^n decays like an **exponential function** $\exp(-\mu t^n)$ for $\mu > 0$, in the full damping case the discrete counterpart of the Timoshenko system is **exponentially stable**.

Nonlinear damping of the type "$|s|s$"

$$\begin{cases} \varphi_{tt} - (\varphi_x + \psi)_x = 0, & (x,t) \in (0,L) \times R_+, \\ \psi_{tt} - b\psi_{xx} + (\varphi_x + \psi) + g(\psi_t) = 0, & (x,t) \in (0,L) \times R_+. \end{cases} \quad (22)$$

$$\begin{cases} M\dfrac{du}{dt} = -K\Phi + S\Psi, \\ M\dfrac{dv}{dt} = -K\Psi - S\Phi - M\Psi - M|V|V, \\ \dfrac{d\Psi}{dt} = V, \\ \dfrac{d\Phi}{dt} = U, \end{cases} \quad (23)$$

where define the term "$|v|v$" as follows

$$V(t)|V(t)| = \sum_{i=0}^{Nx} |v_i(t)|v_i(t)w_i(x). \quad (24)$$

We approximate V^n by $\frac{1}{2}(V^{n+1}+V^{n-1})$.
Find $(\Phi^n, U^n, \Psi^n, V^n)$ such that

$$\begin{cases} M\dfrac{U^{n+1}-U^{n-1}}{2\Delta t} = -K\Phi^n + S\Psi^n, \\ M\dfrac{V^{n+1}-V^{n-1}}{2\Delta t} = -K\Psi^n - S\Phi^n - M\Psi^n - \dfrac{1}{2}M(V^{n+1}+V^{n-1})|V^n|, \\ \dfrac{\Psi^{n+1}-\Psi^{n-1}}{2\Delta t} = V^n, \\ \dfrac{\Phi^{n+1}-\phi^{n-1}}{2\Delta t} = U^n. \end{cases} \tag{25}$$

Theorem 6

For all $\Delta t > 0$ the discrete energy verifies

$$\frac{1}{2\Delta t}\left(E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}}\right) \leq 0.$$

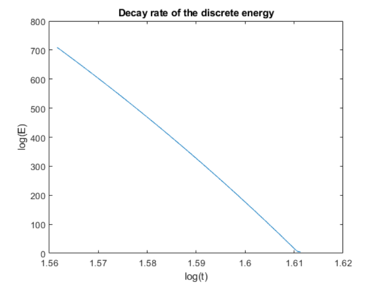


FIGURE – Polynomial decay rate of the discrete energy.

We deduce an explicit decay rate of the energy by giving an approximate value to the polynomial degree :
$\log(E^n) = a_1 \log(t^n) + b_1$, such that $b_1 = 1.61$ and $a_1 = -434.78$.

This example is one of others that has been taken to illustrate the optimal energy decay rate

$$\begin{cases} \varphi_{tt} - (\varphi_x + \psi)_x = 0, & (x,t) \in (0,L) \times R_+, \\ \psi_{tt} - (\psi_{xx} + (\varphi_x + \psi) + g(\psi_t) = 0, & (x,t) \in (0,L) \times R_+. \end{cases} \quad (26)$$

where $g(x) = \exp\left(\frac{-1}{x^2}\right)$.

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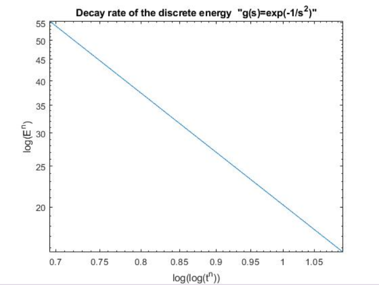

FIGURE – Logarithmic decay rate of discrete energy

Logarithmic decay of the energy. We have $\log(E^n) = a_2 \log(\log(t^n)) + b_2$, such that $b_2 = 1.10$ and $a_2 = -50$ $\rightsquigarrow E^n \simeq e^{b_2}(\log(t^n))^{a_2}$.

Thank you
For your attention !