The Resolution of the Spectral Pollution by the Generalized Spectrum Method.

Ammar KHELLAF

University 08 Mai 1945 (Algeria)

Identification and Control Some new challenges
18-20 Juin 2019, Monastir Tunisia

Contents

Recall and Definitions

- Spectrum of A :

$$\mathrm{sp}(A) := \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \text{ does not exist or is not bounded}\}.$$

- Discrete spectrum :

$$\mathrm{sp}_{\mathrm{p}}(A) := \{ \text{ eigenvalue of } A\}.$$

- Essential spectrum :

$$\mathrm{sp}_{\mathrm{ess}}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is injective but not surjective}\}.$$

- Resolvent set of A :

$$\mathrm{re}(A) = \mathbb{C} \setminus \mathrm{sp}(A).$$

Problematic

Let A be unbounded operator over an Hilbert space $\mathcal{H}$. We define the spectral problem as ; Find $(u, \lambda) \in \mathcal{H} \times \mathbb{C}$ such that

$$Au = \lambda u. \tag{1}$$

The numerical discretization of the equation (1) produces, in some cases, spurious results, a phenomenon called "spectral pollution".

In other words, the spectral pollution is when the approximate matrix may possess eigenvalues which are unrelated to any spectral properties of the original operator of equation (1).

$$
\begin{aligned}
A : D(A) \subset \mathcal{H} &\rightarrow \mathcal{H}, \\
P_k : D(A) &\rightarrow \mathcal{L}_k \subset D(A), \\
\lim_{k \to +\infty} \operatorname{sp}(P_k A P_k) &= \operatorname{sp}(A) \quad ???
\end{aligned}
$$

where the limit is defined in an appropriate sense.

Example

Let $\mathcal{H} = \ell^2(\mathbb{Z})$.
A be the right shift operator : $Ae_n = e_{n+1}$, $n \in \mathbb{Z}$ and let $(P_k)_{k\in\mathbb{N}}$ a sequence of projections such that $P_k\mathcal{H} = \mathcal{L}_k = Span\{e_n\}_{n=-k}^{n=k}$.
Then $\mathrm{sp}(A) = \{z \in \mathbb{C} : |z| = 1\}$, and

$$P_kAP_k = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

But,

$$\mathrm{sp}(P_kAP_k) = \{0\}.$$

$$
\begin{aligned}
T:\mathcal{H} &\rightarrow \mathcal{H}, \\
T_k:\mathcal{H} &\rightarrow \mathcal{H}, \\
\|(T-T_k)\,T_k\| &\rightarrow 0, && (3)\\
\|(T-T_k)\,T\| &\rightarrow 0, && (4)\\
\sup_{k\geq 1}\|T_k\| &< \infty. && (5)
\end{aligned}
$$

Theorem

If, for $k\in\mathbb{N}$, $\lambda_k\in\mathrm{sp}\,(T_k)$ and $\lambda_k\rightarrow\lambda$ then $\lambda\in\mathrm{sp}\,(T)$ (Propriety U)

Theorem

if $\lambda\in\mathrm{sp}(T)$, then there exists $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k\rightarrow\lambda$ and $\lambda_k\in\mathrm{sp}(T_k)$. (Propriety L)

Our vision : Generalized spectrum

Find two bounded operators T, S such that :

$$\begin{aligned} T : \mathcal{H} &\rightarrow \mathcal{H}, \\ S : \mathcal{H} &\rightarrow \mathcal{H}, \\ \mathrm{sp}\,(T, S) &= \mathrm{sp}\,(A). \end{aligned}$$

Then, through a numerical approximation of (T, S) by a sequences of bounded operators $(T_k)_{k\in\mathbb{N}}, (S_k)_{k\in\mathbb{N}}$ converging in a appropriate sense :

$$\begin{aligned} T_k : \mathcal{H} &\rightarrow \mathcal{H}, \\ S_k : \mathcal{H} &\rightarrow \mathcal{H}, \\ \mathrm{sp}\,(T_k, S_k) &\overset{k\to\infty}{\longrightarrow} \mathrm{sp}\,(T, S). \end{aligned}$$

Definitions

- Generalized spectrum of (T, S) :

$$\mathrm{sp}(T,S) := \{\lambda \in \mathbb{C} : (T - \lambda S)^{-1} \text{ does not exist or is not bounded } \}.$$

- Generalized discret spectrum of (T, S) :

$$\mathrm{sp}_{\mathrm{p}}(T,S) := \{\lambda \in \mathbb{C} \ : \ \exists u \in H / \{0\}, \ \ Tu = \lambda S u\}.$$

- Generalized essential spectrum of (T, S) :

$$\mathrm{sp}_{\mathrm{ess}}(T,S) := \{\lambda \in \mathbb{C} : T - \lambda S \text{ is injective but not surjective } \}.$$

- Generalized resolvent set of (T, S) :

$$\mathrm{re}(T,S) := \mathbb{C} / \mathrm{sp}(T,S).$$

Decomposition of unbounded operator to a couple of bounded operators

The following theorem shows that every unbounded operator contains a decomposition of two bounded operators, which express it in the theory of the generalized spectrum (see [4]).

Theorem

Let A be a unbounded operator in $\mathcal{H}$. If $\mathrm{re}(A) \neq \emptyset$, then there exists two bounded operators (T, S) defined on $\mathcal{H}$ such that :

- $\mathrm{sp}(A) = \mathrm{sp}(T, S)$,
- $\mathrm{sp}_p(A) = \mathrm{sp}_p(T, S)$,
- $\forall \lambda \in \mathrm{sp}_p(A)$:

$$Ker(A - \lambda I) = Ker(T - \lambda S).$$

Idea of the proof

We take :

$$\begin{cases} S = (A - \alpha I)^{-1} \\ T = (A - \alpha I)^{-1} A. \end{cases}$$

So,

$$\begin{aligned} \lambda \in re(A) &\Leftrightarrow (A - \lambda I)^{-1} \in \mathrm{BL}(\mathcal{H}) \\ &\Leftrightarrow (A - \lambda I)(A - \alpha I)^{-1} \in \mathrm{BL}(\mathcal{H}) \\ &\Leftrightarrow (T - \lambda S)^{-1} \in \mathrm{BL}(\mathcal{H}) \\ &\Leftrightarrow \lambda \in re(T, S). \end{aligned}$$

$\mathrm{BL}(\mathcal{H})$: is the space of bounded operators given over $\mathcal{H}$.

$$\vdots \quad \vdots \quad \vdots$$

Some Results

Theorem

Let $T, S \in \mathrm{BL}(\mathcal{H})$, *then*

- $\mathrm{sp}(T, S)$ *is an closed set in* $\mathbb{C}$.
- *The function :*

$$\begin{aligned} R(\cdot, T, S) : \mathrm{re}(T, S) &\rightarrow \mathrm{BL}(\mathcal{H}), \\ \lambda &\rightarrow R(\lambda, T, S) = (T - \lambda S)^{-1}, \end{aligned}$$

is an analytical function where its derivative is :
$R(\cdot, T, S)\, S R(\cdot, T, S)$.

These results are naturally extended from the classical case $S = I$. The proofs are established in [1].

Our results in the qualitative plan for the generalized spectrum

Theorem

Let $T, S \in \mathrm{BL}(\mathcal{H})$, *then*

$$\mathrm{sp}(T, S) \subset B(0, k) \iff 0 \notin \mathrm{sp}(S).$$

Theorem

Let $T, S \in \mathrm{BL}(\mathcal{H})$, *If* S *is a compact operator, then*

- $\mathrm{sp}(T, S)$: *is a discret set,*
- $\mathrm{sp}(T, S) = \mathrm{sp}_p(T, S) \cup \{\infty\}$.

The proofs are established in [4].

Generalized spectrum approximation under collectively compact convergence

We say that $\{T_k\}_{k\in\mathbb{N}}$ and $\{S_k\}_{k\in\mathbb{N}}$ are converging to T and S respectively, in the collectively compact convergence if :

$$\lim_{k\to\infty} \|Tx - T_k x\| = 0, \ \lim_{k\to\infty} \|Sx - S_k x\| = 0, \quad \forall x \in \mathcal{H}. \tag{6}$$

and the sets

$$\bigcup_{k\geq k_0} \{Tx - T_k x : \|x\| = 1\}, \quad \bigcup_{k\geq k_0} \{Sx - S_k x : \|x\| = 1\}. \tag{7}$$

are relatively compact.

Theorem

If, for $k \in \mathbb{N}$, $\lambda_k \in \mathrm{sp}\,(T_k, S_k)$ and $\lambda_k \to \lambda$ then $\lambda \in \mathrm{sp}\,(T, S)$. (Propriety U)

Proof of Propriety U under the collectively compact convergence

To prove that Propriety U holds under the collectively compact convergence, we needed to the following lemmas :

Lemma

If $T_k \xrightarrow{p} T$ *and* $S_k \xrightarrow{cc} S$ *then, for any operator* $H \in \mathrm{BL}(X)$*,*

$$\|(T_k - T)H(S_k - S)\| \to 0.$$

Lemma

Let $T, \widetilde{T}, S$ *and* $\widetilde{S}$ *belong to* $\mathrm{BL}(X)$. *Let* $z \in \mathrm{re}(T, S)$. *If*

$$\left\| \left[\left((T - \widetilde{T}) - z(S - \widetilde{S}) \right) R(z, T, S) \right]^2 \right\| < 1,$$

then $z \in \mathrm{re}(\widetilde{T}, \widetilde{S})$, *and*

$$\|(\widetilde{T} - z\widetilde{S})^{-1}\| \leq \frac{\|R(z, T, S)\| \left[1 + \| \left((T - \widetilde{T}) - z(S - \widetilde{S}) \right) R(z, T, S) \| \right]}{1 - \left\| \left[\left((T - \widetilde{T}) - z(S - \widetilde{S}) \right) R(z, T, S) \right]^2 \right\|}.$$

Proof

Suppose that $\lambda \in \mathrm{re}(T, S)$. Since the set $\mathrm{re}(T, S)$ is open in $\mathbb{C}$, there exists $r > 0$ such that

$$B = \{z \in \mathbb{C} : \quad |\lambda - z| \leq r\} \subset \mathrm{re}(T, S).$$

Since $R(T, S, \cdot)$ is analytic on $\mathrm{re}(T, S)$,

$$\alpha = \sup \{\|R(T, S, z)\| : \quad z \in B \} < +\infty$$

According to Lemma 8, for all $z \in B$, there exist k_1, k_2, k_3 and k_4 such that

$$\|(T - T_k)R(T,S,z)(T - T_k)\| \leq \frac{1}{8\alpha} \text{ for } k \geq k_1,$$

$$\|(S - S_k)R(T,S,z)(S - S_k)\| \leq \frac{1}{8(r + |z|^2)\alpha} \text{ for } k \geq k_2,$$

$$\|(T - T_k)R(T,S,z)(S - S_k)\| \leq \frac{1}{8(r + |z|)\alpha} \text{ for } k \geq k_3,$$

$$\|(S - S_k)R(T,S,z)(T - T_k)\| \leq \frac{1}{8(r + |z|)\alpha} \text{ for } k \geq k_4.$$

Then, for $k \geq \max\{k_1, k_2, k_3, k_4\}$,

$$\| [((T - T_k) - z(S - S_k))R(T,S,z)]^2 \| \leq [\ \|(T - T_k)R(T,S,z)(T - T_k)\|$$

$$+|z|^2\|(S - S_k)R(T,S,z)(S - S_k)\| + |z|\|(T - T_k)R(T,S,z)(S - S_k)\|$$
$$+|z|\|(S - S_k)R(T,S,z)(T - T_k)\|\ ]\ \|R(T,S,z)\|$$
$$\leq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}.$$

Using lemma 9, $(T_k - zS_k)^{-1}$ exists and is bounded. Therefore $B \subset \mathrm{re}(T_k, S_k)$ for all $k \in \mathbb{N}$ large enough, which is contradictory with the hypothesis $\lambda_k \in \mathrm{sp}(T_k, S_k)$ for all $k \in \mathbb{N}$. Hence $\lambda \in \mathrm{sp}(T, S)$.

∎

In numerical test, the quantity

$$\sup\left\{ dist\left(\mu, \mathrm{sp}\left(T, S\right)\right) \ : \ \mu \in \mathrm{sp}\left(T_k, S_k\right)\right\}, \tag{8}$$

can be computed because, if it converges to 0, then Propriety U holds.

Definition

Let λ be an isolated point of $\mathrm{sp}(T, S)$. We say that λ has a finite algebraic multiplicity if there exists $\ell \in \mathbb{N}$ such that

$$\alpha = \dim \mathbb{K}\mathrm{er}(T - \lambda S)^{\ell} < \infty.$$

α is the generalized geometric multiplicity.

If $\alpha = 1$, λ is called a simple generalized eigenvalue of finite type.

The propriety L shows that, if $\lambda \in sp(T, S)$, then there exists $\lambda_k \in sp(T_k, S_k)$ such that : $\lambda_k \to \lambda$.

Theorem

Let be $T, S \in \mathrm{BL}(\mathcal{H})$, and $\lambda \in \mathrm{sp}(T, S)$ be an isolated generalized eigenvalue of finite type. Let Γ be a Cauchy contour separating λ from $\mathrm{sp}(T, S)$. Then the operator

$$P = -\frac{1}{2i\pi} \int_{\Gamma} (T - zS)^{-1} S \, dz,$$

defines a projection.

This is a classical result established in [1].

Theorem

Under the same hypotheses as in the above theorem, if T commutes with S, then

$$PX = \mathbb{K}er(T - \lambda S)^{\ell},$$

where $\ell \geq 1$.

This is a generalized result for the spectral decomposition theory established for the cases $S = I$.

We assume that $\{T_k\}_{k\in\mathbb{N}}$ and $\{S_k\}_{k\in\mathbb{N}}$ are converging to T and S respectively, in the collectively compact convergence.

Theorem

Under the same hypotheses as in the above theorem, there existe k_0 such that

$$\dim(PX) = \dim(P_k X),$$

for $k \geq k_0$, where

$$P_k = \frac{-1}{2i\pi}\int_\Gamma (T_k - zS_k)^{-1} S_k \, dz.$$

Ideas of the proof

To prove the last statement, we show that

$$\lim_{k\to\infty} \|(P - P_k)P\| = 0,$$

and

$$\lim_{k\to\infty} \|(P - P_k)P_k\| = 0,$$

Then we use the next lemma.

Lemma

Let P_1 and P_2 be projections on X such that $\|(P_1 - P_2)P_1\| < 1$. Then

$$\dim P_1 X \leq \dim P_2 X.$$

Propriety L

The next theorem shows that the propriety L is hold.

Theorem

Let $\lambda \in \mathrm{sp}(T, S)$ *be an isolated generalized eigenvalue of finite type. Under the collectively compact convergence, there exists a sequence* $\lambda_k \in \mathrm{sp}(T_k, S_k)$ *such that* $\lambda_k \to \lambda$.

Démonstration.

Let Γ be a Cauchy contour isolating λ from $\mathrm{sp}(T,S)$. We set

$$\lambda_k = \mathrm{int}(\Gamma) \cap \mathrm{sp}(T_k, S_k).$$

Since $\mathrm{re}(T,S) \ni z \mapsto (T - zS)^{-1}S$ and $\mathrm{re}(T_k, S_k) \ni z \mapsto (T_k - zS_k)^{-1}S_k$ are analytic functions, and since $P_k \xrightarrow{p} P$, we find

$$\{\lambda_k : k \in \mathbb{N}\} = \emptyset \Longleftrightarrow \mathrm{int}(\Gamma) \cap \mathrm{sp}(T,S) = \emptyset.$$

We fix $\epsilon > 0$ such that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ lies in

$$E := \{z \in \mathbb{C} : |z - \lambda| \leq \epsilon\}.$$

It is enough to show that any convergent subsequence of $(\lambda_k)_{k\in\mathbb{N}}$ converges to λ. Let $(\lambda_{k'})_{k'\in\mathbb{N}}$ a subsequence that converges to $\tilde{\lambda} \neq \lambda$. By Propriety U, $\tilde{\lambda} \in \mathrm{sp}(T,S)$. But $\tilde{\lambda} \in E$ and $\mathrm{sp}(T,S) \cap E := \{\lambda\}$. Hence $\lambda = \tilde{\lambda}$, thus $\lambda_k \to \lambda$. □

Let $q \in L^1_{loc}(0, +\infty)$ be a real positive function, and let ϑ be a fixed number in $[0, \pi)$. We denote by B^ϑ the sectorial operator (see [2]) defined by

$$B^\vartheta u = -u'' + qu, \tag{9}$$

where,

$$D(B^\vartheta) = \left\{u \in H^2(0, +\infty) : \cos\vartheta\, u(0) - \sin\vartheta\, u'(0) = 0\right\} \cap Y,$$

$$Y = \{u \in L^2(0, +\infty) : \int_0^{+\infty} q(x)|u(x)|^2 dx < \infty\}.$$

Similarly, for all $a > 0$, we define the sectorial operator B_a^{ϑ} by :

$$B_a^{\vartheta} u = -u'' + qu, \tag{10}$$

where,

$$D(B_a^{\vartheta}) = \{ u \in H^2(0,a) : \cos\vartheta\, u(0) - \sin\vartheta\, u'(0) = 0,\ \ u(a) = 0 \ \} \cap Y_a,$$

$$Y_a = \{ u \in L^2(0,a) : \int_0^a q(x)|u(x)|^2 dx < \infty \}.$$

Theorem

$$\mathrm{sp}(B^{\vartheta}) = \bigcup_{a>0} \mathrm{sp}(B_a^{\vartheta}).$$

This result is given in [2].

Generalized spectrum

Let $G_{0,a}$ be the Green kernel :

$$G_{0,a}(x,y) = \begin{cases} \dfrac{(\cos\vartheta x - \sin\vartheta)(a-y)}{a\cos\vartheta + \sin\vartheta} & \text{for } 0 \le x \le y \le a, \\ \dfrac{(\cos\vartheta y - \sin\vartheta)(a-x)}{a\cos\vartheta + \sin\vartheta} & \text{for } 0 \le y \le x \le a. \end{cases}$$

Let T, S be two the bounded operators defined on $L^2(0,a)$ to itself by

$$Su(x) = \int_0^a G_{0,a}(x,y)u(y)dy.$$

$$Tu(x) = u(x) + \int_0^a G_{0,a}(x,y)q(y)u(y)dy.$$

Theorem

$$\mathrm{sp}(B_a^{\vartheta}) = \mathrm{sp}(T, S).$$

This result is given in [2].

We fixe $a = 5$, and $q(x) = x^2$, so we are dealing with the harmonic operator.

$$Tu(x) = u(x) + \int_0^a G_{0,a}(x,y)y^2u(y)dy, \;\; Su(x) = \int_0^a G_{0,a}(x,y)u(y)dy,$$

and

$$G_{0,a}(x,y) = \begin{cases} \dfrac{x(a-y)}{a} \text{ for } 0 \leq x \leq y \leq a, \\ \\ \dfrac{y(a-x)}{a} \text{ for } 0 \leq y \leq x \leq a. \end{cases}$$

Three numerical methods will be used to approach the operators T and

S : Nyström's, Sloan's and Kantorovich's methods.

Nyström's method

We define a uniform grid in $[0, a]$. For $n \geq 2$,

$$h_n = \frac{a}{n-1}, \; x_i = (i-1)h_n, \; 1 \leq i \leq n.$$

Let T_n and S_n be approximations of T and S respectively built by the Nyström method :

$$\begin{aligned} T_n u_n(x) &= u_n(x) + \sum_{i=1}^{n} w_i G_{0,a}(x, y_i) y_i^2 u_n(y_i), \\ S_n u_n(x) &= \sum_{i=1}^{n} w_i G_{0,a}(x, y_i) u_n(y_i), \end{aligned}$$

where, $\{w_i\}_{i=1}^{n}$ are real weights such that,

$$\sup_{n \geq 2} \sum_{i=1}^{n} |w_i| < \infty.$$

The corresponding auxiliary matrix generalized eigenvalue problem corresponds to the pencil $A - \lambda_n B$, where

$$A(i,j) = I(i,j) + w_i G_{0,a}(x_j, y_i) y_i^2, \quad B(i,j) = w_i G_{0,a}(x_j, y_i),$$

and I denotes the identity matrix of order n. We use the function "eig" in Matlab to compute the generalized eigenvalues of (A, B).

Sloan's method

With the previous uniform grid, we define the approximate operators $\widetilde{T}_n$ and $\widetilde{S}_n$ by Sloan's method :

$$\begin{aligned} \widetilde{T}_n u_n(x) &= \sum_{i=1}^{n} u_n(x_i) e_i(x) + \sum_{i=1}^{n} w_{1,i}(x) u_n(x_i), \\ \widetilde{S}_n u_n(x) &= \sum_{i=1}^{n} w_{2,i}(x) u_n(x_i), \end{aligned}$$

where,

$$w_{1,i}(x) = \int_0^a G_{0,a}(x,y) y^2 e_i(y) dy, \; w_{2,i}(x) = \int_0^a G_{0,a}(x,y) e_i(y) dy.$$

for $1 \leq i \leq n$.

for $2 \leq i \leq n-1$,

$$e_i(x) = \begin{cases} 1 - \dfrac{|x - x_i|}{h_n} & \text{for } x_{i-1} \leq x \leq x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

$$e_1(x) = \begin{cases} \dfrac{x_2 - x}{h_n} & \text{for } x_1 \leq x \leq x_2, \\ 0 & \text{otherwise.} \end{cases}$$

$$e_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{h_n} & \text{for } x_{n-1} \leq x \leq x_n, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding auxiliary matrix generalized eigenvalue problem corresponds to the pencil $\widetilde{A} - \lambda_n \widetilde{B}$, where

$$\widetilde{A}(i,j) = I(i,j) + w_{1,i}(x_j), \;\; \widetilde{B}(i,j) = w_{2,i}(x_j).$$

We use the function "eig" in Matlab to compute the generalized eigenvalue of $(\widetilde{A}, \widetilde{B})$.

The Kantorovich method

With the same uniform grid as before, we apply Kantorovich's projection method, we get for all $x \in [0, a]$

$$u_n(x) + \sum_{i=1}^{n} \left(\int_0^a G_{0,a}(x_i, y) y^2 u_n(y) dy \right) e_i(x)$$
$$= \lambda_n \sum_{i=1}^{n} \left(\int_0^a G_{0,a}(x_i, y) u_n(y) dy \right) e_i(x).$$

Multiplying first by $G_{0,a}(x_j, x)x^2$ then by $G_{0,a}(x_j, x)$, and integrating over $[0, a]$, this equation leads to the following matrix generalized eigenvalue problem :

$$\begin{bmatrix} \widetilde{A} + I_{n\times n} & O_{n\times n} \\ \widetilde{B} & I_{n\times n} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \lambda_{2n} \begin{bmatrix} O_{n\times n} & \widetilde{A} \\ O_{n\times n} & \widetilde{B} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

where,

$$\beta_1(j) = \int_0^a G_{0,a}(x_j, y) y^2 u_n(y) dy, \ \beta_2(j) = \int_0^a G_{0,a}(x_j, y) u_n(y) dy, \ 1 \leq j \leq n.$$

Matrices $\widetilde{A}$ and $\widetilde{B}$ are the same as those of Sloan's case.
As before, we use the function "eig" in Matlab to compute the generalized eigenvalues of the pencil

$$\begin{bmatrix} \widetilde{A} + I_{n\times n} & O_{n\times n} \\ \widetilde{B} & I_{n\times n} \end{bmatrix} - \lambda_n \begin{bmatrix} O_{n\times n} & \widetilde{A} \\ O_{n\times n} & \widetilde{B} \end{bmatrix}.$$

We have chosen $n = 200$ to exhibit the numerical results obtained by each of these three approximations. Results are compared with those of [3] in the following tables.

Table – The numerical results for a=5

Exact eigenvalue	Nyström for T and S	Sloan for T and S	Kantorovich for T and S
3	2.9998027	3.0001972	2.9621125
7	6.9990159	7.0009887	6.8083144
11	10.9977898	11.0026039	10.5272610
15	15.0013776	15.0103317	14.1401140
19	19.0656824	19.0806050	17.8348945

Table – The numerical results for a=5

Exact eigenvalue	Nyström for T Sloan for S	Sloan for T Nyström for S	[7]
3	3.0002761	2.9997238	2.9621125
7	7.0015938	6.9984110	6.8083144
11	11.0041582	10.9962364	10.5272610
15	15.0132972	14.9984149	14.1401140
19	19.0857154	19.0605807	17.8348945

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Merci Pour Votre Attention.