Problème de Calderón

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This approach, initiated by **Sylvester and Uhlmann 1987**, is one of the main for solving this inverse problem. In this section we will use several arguments borrowed from several references.

Complex geometric optics solutions

The complex geometric optics solutions under consideration here will depend on $\xi \in \mathbb{R}^n$. More precisely, for each $\xi \in \mathbb{R}^n$, we fix $\eta_1, \eta_2 \in \mathbb{S}^{n-1}$ such that

$$\xi \cdot \eta_1 = \xi \cdot \eta_2 = \eta_1 \cdot \eta_2 = 0.$$

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Then, for $\rho > 1$ a large parameter, we consider solutions of $-\Delta v_j + q_j v_j = 0$ on Ω taking the form

$$v_1(x) = e^{\rho\eta_1 \cdot x} \left(e^{i\rho\eta_2 \cdot x} e^{-i\xi \cdot x} + w_1(x) \right), \ x \in \Omega, \tag{1}$$

$$v_2(x) = e^{-\rho\eta_1 \cdot x} \left(e^{-i\rho\eta_2 \cdot x} + w_2(x) \right), \ x \in \Omega, \tag{2}$$

with $w_j \in H^2(\Omega)$, $j = 1, 2$, satisfying

$$\|w_j\|_{H^k(\Omega)} = \underset{\rho \to +\infty}{\mathcal{O}} \left(\rho^{k-1} \right), \ j = 1, 2, \ k = 0, 2. \tag{3}$$

Complex geometric optics solutions

Proposition 1

For $j = 1, 2$ and for all $\xi \in \mathbb{R}^n$, there exists $\rho_1 > 0$ such that for all $\rho > \rho_1$ we can find a solution v_j of $-\Delta v_j + q_j v_j = 0$ on Ω taking the form (1)-(2) with $w_j \in H^2(\Omega)$ satisfying (3).

Proof of Theorem 2 by admitting Proposition 1

We will complete the proof of Theorem 2 by admitting Proposition 1. For this purpose, let us assume that $\Lambda_{q_1} = \Lambda_{q_2}$. We fix $\xi \in \mathbb{R}^n$ and applying Proposition 1 we deduce the existence of $v_j \in H^2(\Omega)$, $j = 1, 2$, solving $-\Delta v_j + q_j v_j = 0$ of the form (1)-(2) with $w_j \in H^2(\Omega)$ satisfying (3).

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Consider $y_2 \in H^2(\Omega)$ solving

$$\begin{cases} -\Delta y_2 + q_2 y_2 = 0 & \text{in } \Omega, \\ y_2 = v_1 & \text{on } \partial\Omega. \end{cases}$$

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$$\begin{cases} -\Delta y_2 + q_2 y_2 = 0 & \text{in } \Omega, \\ y_2 = v_1 & \text{on } \partial\Omega. \end{cases}$$

Then, $v = y_2 - v_1$ solves

$$\begin{cases} -\Delta v + q_2 v = (q_1 - q_2) v_1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof of Theorem 2 by admitting Proposition 1

Moreover, fixing $\varphi \in H^{\frac{3}{2}}(\partial\Omega)$ defined by

$$\varphi(x) = v_1(x), \quad x \in \partial\Omega,$$

we find

$$\partial_\nu v = \partial_\nu y_2 - \partial_\nu v_1 = \Lambda_{q_2}\varphi - \Lambda_{q_1}\varphi = 0.$$

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Thus, v satisfies the condition

$$\begin{cases} -\Delta v + q_2 v = (q_1 - q_2)v_1 & \text{in } \Omega, \\ v = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof of Theorem 2 by admitting Proposition 1

Multiplying, by v_2 and integrating by parts we get the orthogonality identity

$$\int_\Omega (q_1 - q_2) v_1 v_2 dx = 0. \tag{4}$$

In addition, (1)-(2) imply

$$\int_\Omega (q_1 - q_2) v_1 v_2 dx = \int_\Omega (q_1 - q_2) e^{-ix\cdot\xi} dx + \underset{\rho\to+\infty}{o}(1).$$

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Fixing $q = q_1 - q_2$ on Ω, $q = 0$ on $\mathbb{R}^n \setminus \Omega$ and sending $\rho \to +\infty$ we get

$$\int_{\mathbb{R}^n} q(x) e^{-ix\cdot\xi} dx = 0.$$

Since this identity is true for any $\xi \in \mathbb{R}^n$, we deduce that $q = 0$ and $q_1 = q_2$.

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This proves Theorem 2. **It remains to prove Proposition 1 ! ! !**

Proof of Proposition 1

Since the proof of this result is similar for v_1 and v_2, we will only consider the construction of v_1. Note first that

$$\Delta\left(e^{\rho\eta_1\cdot x}e^{i\rho\eta_2\cdot x}\right)=\rho^2\left(|\eta_1|^2-|\eta_2|^2\right)e^{\rho\eta_1\cdot x}e^{i\rho\eta_2\cdot x}=0.$$

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Therefore, we have

$$-\Delta v_1=e^{\rho\eta_1\cdot x}\left(e^{i\rho\eta_2\cdot x}|\xi|^2e^{-i\xi\cdot x}\right)+e^{\rho\eta_1\cdot x}(-\Delta w_1-2\rho\eta_1\cdot\nabla w_1-\rho^2 w_1).$$

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It follows

$$\begin{aligned}&-\Delta v_1+q_1v_1=0\\ \iff&-\Delta w_1-2\rho\eta_1\cdot\nabla w_1-\rho^2w_1+q_1w_1=-e^{i\rho\eta_2\cdot x}(|\xi|^2+q_1)e^{-i\xi\cdot x}.\end{aligned}$$

Proof of Proposition 1

We need to build $w_1 \in H^2(\Omega)$ satisfying

$$-\Delta w_1 - 2\rho\eta_1 \cdot \nabla w_1 - \rho^2 w_1 + q_1 w_1 = -e^{i\rho\eta_2 \cdot x}(|\xi|^2 + q_1)e^{-i\xi \cdot x} \qquad (5)$$

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For this purpose, we fix S an isometric operator of $\mathbb{R}^n$ such that $S\eta_1 = e_1 = (1, 0, \ldots, 0)$ and $R > 0$ such that $\Omega \subset Q := S^*(-R, R)^n$. Here S^* denotes the adjoint of S with respect to the Euclidean scalar product of $\mathbb{R}^n$.

Proof of Proposition 1

Then, for $F \in L^2(Q)$ we consider a solution $z \in H^2(Q)$ of

$$-\Delta z - 2\rho\eta_1 \cdot \nabla z - \rho^2 z = F, \quad \text{on } Q \tag{6}$$

taking the form

$$z(x) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \varphi_\alpha, \quad \varphi_\alpha(x) := (2R)^{-\frac{n}{2}} e^{\frac{i\pi\eta_1 \cdot x}{2R}} e^{\frac{i\pi\alpha \cdot S(x)}{R}}. \tag{7}$$

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The construction of such solutions can be deduced from the following result (based on a simplified version of **Hähner 1996**) that we admit for the moment.

Lemma 1

For all $F \in L^2(Q)$ the equation (6) *admits a solution $z_F \in H^2(Q)$ taking the form* (7) *and satisfying*

$$\|z_F\|_{H^k(Q)} \leqslant C\rho^{k-1} \|F\|_{L^2(Q)}, \quad k = 0, 2, \tag{8}$$

with C independent of F and ρ.

Proof of Proposition 1

Applying Lemma 1, we can define the operator

$$\mathcal{K}_\rho : L^2(Q) \ni F \mapsto z_F|_\Omega \in H^2(\Omega)$$

which is bounded. Here z_F denotes a solution of (6) satisfying (8). We have $\mathcal{K}_\rho \in \mathcal{B}(L^2(Q); H^2(\Omega))$ and

$$\|\mathcal{K}_\rho\|_{\mathcal{B}(L^2(Q);H^k(\Omega))} \leqslant C\rho^{k-1}, \quad k = 0, 2. \tag{9}$$

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Fix $F_\rho \in L^2(Q)$ and $\widetilde{q}_1 \in L^\infty(Q)$ defined by

$$F_\rho(x) := \begin{cases} -e^{i\rho\eta_2\cdot x}(|\xi|^2 + q_1(x))e^{-i\xi\cdot x} & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

$$\widetilde{q}_1(x) := \begin{cases} q_1(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Proof of Proposition 1

Now consider the map

$$\begin{aligned}\mathcal{G}_\rho : L^2(\Omega) &\to L^2(\Omega),\\ w &\mapsto \mathcal{K}_\rho\left[F_\rho - \widetilde{q}_1 w\right].\end{aligned}$$

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We fix $B = \{h \in L^2(\Omega) : \ \|h\|_{L^2(\Omega)} \leqslant 1\}$ of $L^2(\Omega)$ and we will prove that there exists $\rho_1 > 0$ such that, for $\rho > \rho_1$, $\mathcal{G}_\rho$ admits a unique fixed point $w_1 \in B$.

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$$\begin{aligned}\forall y_1, y_2 \in B, \quad \|\mathcal{G}_\rho y_1 - \mathcal{G}_\rho y_2\|_{L^2(\Omega)} &= \|\mathcal{K}_\rho\left[\widetilde{q}_1(y_2 - y_1)\right]\|_{L^2(\Omega)}\\ &\leqslant C\rho^{-1}\|q_1\|_{L^\infty(\Omega)}\|y_1 - y_2\|_{L^2(\Omega)}\end{aligned}$$

and

$$\forall y \in B, \quad \|\mathcal{G}_\rho y\|_{L^2(\Omega)} \leqslant C\rho^{-1}(2\|q_1\|_{L^\infty(\Omega)} + |\Omega|^{\frac{1}{2}}|\xi|^2).$$

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Therefore, there exists $\rho_1 > 1$ such that for $\rho > \rho_1$, $\mathcal{G}_\rho$ admits a unique fixed point w_1 in B.

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Moreover, we have $w_1 = \mathcal{K}_\rho \left[F_\rho - \widetilde{q}_1 w_1\right]$ which proves that $w_1 \in H^2(\Omega)$ solves

$$\begin{aligned} &[-\Delta w_1 - 2\rho\eta_1 \cdot \nabla w_1 - \rho^2 w_1](x) \\ &= F_\rho(x) - \widetilde{q}_1 w_1(x) \\ &= -e^{i\rho\eta_2\cdot x}(|\xi|^2 + q_1(x))e^{-i\xi\cdot x} - q_1(x)w_1(x), \quad x \in \Omega. \end{aligned}$$

Therefore, w_1 solves (5) and (9) implies

$$\|w_1\|_{H^k(\Omega)} \leqslant C\rho^{k-1}(2\|q_1\|_{L^\infty(\Omega)} + |\Omega|^{\frac{1}{2}}|\xi|^2),\ k = 0, 2,$$

which proves (3). □