Problème de Calderón

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Proof of Lemma 1

Then, for $F \in L^2(Q)$ we consider a solution $z \in H^2(Q)$ of

$$-\Delta z - 2\rho\eta_1 \cdot \nabla z - \rho^2 z = F, \quad \text{on } Q \tag{1}$$

taking the form

$$z(x) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \varphi_\alpha, \quad \varphi_\alpha(x) := (2R)^{-\frac{n}{2}} e^{\frac{i\pi\eta_1 \cdot x}{2R}} e^{\frac{i\pi\alpha \cdot S(x)}{R}}. \tag{2}$$

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The construction of such solutions can be deduced from the following result (based on a simplified version of **Hähner 1996**) that we admit for the moment.

Lemma 1

For all $F \in L^2(Q)$ the equation (1) admits a solution $z_F \in H^2(Q)$ taking the form (2) and satisfying

$$\|z_F\|_{H^k(Q)} \leqslant C\rho^{k-1} \|F\|_{L^2(Q)}, \quad k = 0, 2, \tag{3}$$

with C independent of F and ρ.

PProof of Lemma 1

For $\alpha \in \mathbb{Z}^n$, we fix

$$a_\alpha := \frac{\langle F, \varphi_\alpha \rangle_{L^2(Q)}}{\frac{\pi^2}{R^2} \left| S^* \alpha + \frac{\eta_1}{2} \right|^2 - \rho^2 - \frac{2 i \pi \rho}{R} \left(S^* \alpha \cdot \eta_1 + \frac{1}{2} \right)}.$$

We have seen that with this choice z solves

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It remains to show that $z \in H^2(Q)$ and z satisfies (3). For this purpose, we recall that

$$\begin{aligned}&\left|\frac{\pi^2}{R^2}\left|S^*\alpha + \frac{\eta_1}{2}\right|^2 - \rho^2 - \frac{2i\pi\rho}{R}\left(S^*\alpha \cdot \eta_1 + \frac{1}{2}\right)\right| \\ &\geqslant \max\left(\left|\frac{\pi^2}{2R^2}|\alpha|^2 - \frac{\pi^2}{4R^2} - \rho^2\right|, \frac{\pi\rho}{R}\right).\end{aligned} \tag{4}$$

Proof of Lemma 1

Applying (4), we get

$$\sum_{\alpha\in\mathbb{Z}^n}(1+|\alpha|^2)^2|a_\alpha|^2$$

$$\leqslant \sum_{|\alpha|^2\geqslant 2+\frac{4R^2\rho^2}{\pi^2}}(1+|\alpha|^2)^2|a_\alpha|^2+\sum_{|\alpha|^2\leqslant 2+\frac{4R^2\rho^2}{\pi^2}}(1+|\alpha|^2)^2|a_\alpha|^2$$

$$\leqslant \sum_{|\alpha|^2\geqslant 2+\frac{4R^2\rho^2}{\pi^2}}\frac{(1+|\alpha|^2)^2\left|\langle F,\varphi_\alpha\rangle_{L^2(Q)}\right|^2}{\left|\frac{\pi^2}{2R^2}\,|\alpha|^2\right|^2}+\sum_{|\alpha|^2\leqslant 2+\frac{4R^2\rho^2}{\pi^2}}\frac{(1+|\alpha|^2)^2\left|\langle F,\varphi_\alpha\rangle_{L^2(Q)}\right|^2}{\frac{\pi^2\rho^2}{R^2}}$$

$$\leqslant \sum_{|\alpha|^2\geqslant 2+\frac{4R^2\rho^2}{\pi^2}}\frac{16R^4}{\pi^4}\left|\langle F,\varphi_\alpha\rangle_{L^2(Q)}\right|^2+\frac{\left(3+\frac{4R^2\rho^2}{\pi^2}\right)^2}{\frac{\pi^2\rho^2}{R^2}}\sum_{|\alpha|^2\leqslant 2+\frac{4R^2\rho^2}{\pi^2}}\left|\langle F,\varphi_\alpha\rangle_{L^2(Q)}\right|^2$$

$$\leqslant C\rho^2\sum_{\alpha\in\mathbb{Z}^n}\left|\langle F,\varphi_\alpha\rangle_{L^2(Q)}\right|^2=C\rho^2\,\|F\|^2_{L^2(Q)}\,.$$

Proof of Lemma 1

This proves that $z \in H^2(Q)$ fulfills (3) for $k = 2$ corresponding to

$$\|z\|_{H^2(Q)} \leqslant C\rho \|F\|_{L^2(Q)} .$$

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In the same way, using (4), we obtain

$$|a_\alpha| \leqslant \frac{R \left| \langle F, \varphi_\alpha \rangle_{L^2(Q)} \right|}{\pi \rho}$$

and we get

$$\|z\|_{L^2(Q)} \leqslant C\rho^{-1} \|F\|_{L^2(Q)} ,$$

with C independent of F and ρ. This proves (3) for $k = 0$ and it completes the proof of the lemma.