Simultaneous and indirect control of waves: some recent developments and open problems

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Course outline

Day 1: General abstract model and simultaneous control

Day 2: Indirect control of two coupled systems

Day 3: Open problems

- An abstract model

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    - Brief literature

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- Simultaneous control
  - Brief literature
  - Lamé systems with localized damping

Model formulation

Let H and V be Hilbert spaces with $V \subset H$. Assume V is dense in H and the injection of V into H is compact. Denote by $(.,.)$ the inner product in H, by $|.|$ the corresponding norm, and by V' the dual of V. Consider the damped abstract equation

$$
\begin{array}{l}
y_{tt} + Ay + By_t = 0 \text{ in } (0, \infty) \\
y(0) = y^0 \in V, \quad y_t(0) = y^1 \in H,
\end{array}
$$

where $A \in \mathcal{L}(V, V')$ is a selfadjoint coercive operator with $D(A^{\frac{1}{2}}) = V$, and $B \in \mathcal{L}(H)$ is a nonnegative operator.

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where $A \in \mathcal{L}(V, V')$ is a selfadjoint coercive operator with $D(A^{\frac{1}{2}}) = V$, and $B \in \mathcal{L}(H)$ is a nonnegative operator.

Introduce the energy

$$E(t) = \frac{1}{2}\{|y_t(t)|^2 + |A^{\frac{1}{2}}y(t)|^2\}, \quad \forall t \geq 0.$$

Theorem: Dafermos criterion

1970: Dafermos proves: the abstract system is strongly stable

$$\lim_{t \to \infty} E(t) = 0$$

if and only if

$$\mathrm{Ker} B \cap \mathrm{Ker}(A + \lambda I) = \{0\}, \quad \forall \lambda \in \mathbb{R}$$

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For the stabilization of single component systems, we refer to the contributions of Bardos-Lebeau-Rauch, Rauch-Taylor, Russell, Dafermos, Chen, Haraux, Komornik, Lasiecka, Nakao, Liu, Martinez, Triggiani, Zuazua,...

Brief literature

By simultaneous stabilization, we should understand stabilizing a multi-component system using the same damping mechanism in all components; the matrix defining the damping is degenerate.

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- 1986: Russell introduces the notion of simultaneous control for pdes when studying the boundary controllability of the Maxwell's equations.
- 1988: Lions (v.1, Controllability book) analyzes simultaneous boundary control problems for two uncoupled waves, and for two uncoupled plates.

Brief literature

Consider the system of uncoupled wave equations

$$\begin{aligned}
&u_{jtt} - a_j \Delta u_j = 0 \text{ in } Q \\
&u_j = 0 \text{ on } \Gamma \times (0, T) \\
&u_j(x,0) = u_j^0(x), \quad u_{jt}(x,0) = u_j^1(x) \text{ in } \Omega, \quad j = 1,\ 2, ...,\ q,
\end{aligned}$$

where $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$ for each j.

1988: Haraux (1988) shows for arbitrary nonempty open set ω:

- If $\sum_{j=1}^{q} u_j(x,t) = 0$ in $\omega \times (0,T)$ then $u_j^0 = 0, \quad u_j^1 = 0$ in $\Omega, \quad \forall j$.
  provided that $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.

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- If $N = 1$ and T is large enough, or $\omega = \Omega$, then there exists $C > 0$: for all j and all $(u_j^0, u_j^1) \in L^2(\Omega) \times H^{-1}(\Omega)$

$$\sum_{j=1}^{q} \{||u_j^0||^2_{L^2(\Omega)} + ||u_j^1||^2_{H^{-1}(\Omega)}\} \leq C \int_0^T \int_\omega |\sum_{j=1}^{q} u_j(x,t)|^2 \, dxdt$$

  provided that $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.

(GCC): Bardos-Lebeau-Rauch (1992): ω is an admissible control region in time T if every ray of geometric optics enters ω in a time less than T.

Theorem 1 (CRAS, Paris, 2012)

Let T_0 denote the best controllability time for a single wave equation with unit speed of propagation. Suppose that $T > T_0 \max\{a_j^{-\frac{1}{2}}; j = 1,\ 2,\ ...,\ q\}$ and (ω, T) satisfies (GCC). There exists a constant $C > 0$ such that for all $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $j = 1,\ 2, ...,\ q$:

$$\sum_{j=1}^{q} \{||u_j^0||^2_{H_0^1(\Omega)} + ||u_j^1||^2_{L^2(\Omega)}\} \leq C \int_0^T \int_\omega |\sum_{j=1}^{q} u_{jt}(x,t)|^2\, dxdt,$$

with $C = C(\Omega, \omega, T, (a_j)_j, q)$, if and only if $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.

Lamé systems with localized damping

Given $(y_j^0, y_j^1)_j \in \left(\left[H_0^1(\Omega) \right]^N \times \left[L^2(\Omega) \right]^N \right)^q$, and a function $d \in L^\infty(\Omega)$, $d \geq 0$, consider the damped elastodynamic system

$$
\begin{aligned}
& y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j) \nabla \mathrm{div}(y_j) + d \sum_{k=1}^{q} y_{kt} = 0 \text{ in } \Omega \times (0, \infty) \\
& y_j = 0 \text{ on } \Gamma \times (0, \infty) \\
& y_j(x, 0) = y_j^0(x), \quad y_{jt}(x, 0) = y_j^1(x), \text{ in } \Omega, \\
& j = 1,\ 2, \ldots,\ q,
\end{aligned}
$$

where, for each j, μ_j and λ_j are the Lamé constants.
The total energy is given, for all $t \geq 0$, by

$$
2E(t) = \sum_{j=1}^{q} \int_\Omega \{ |y_{jt}(x,t)|^2 + \mu_j |\nabla y_j(x,t)|^2 + (\mu_j + \lambda_j) |\mathrm{div}(y_j(x,t))|^2 \} \, dx
$$

Lamé systems with localized damping

E is a nonincreasing function of the time variable as

$$\frac{dE}{dt}(t) = -\int_{\Omega} d(x) \left| \sum_{k=1}^{q} y_{kt}(x,t) \right|^2 dx.$$

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Question 1: Does the energy E decay to zero as time goes to infinity?

Lamé systems with localized damping

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Question 1: Does the energy E decay to zero as time goes to infinity?
Question 2: Under which conditions is the Lamé system exponentially stable?

Introduce the Hilbert space $\mathcal{H} = \left(\left[H_0^1(\Omega) \right]^N \times \left[L^2(\Omega) \right]^N \right)^q$ over the field $\mathbb{C}$ of complex numbers, equipped with the norm

$$||Z||_{\mathcal{H}}^2 = \sum_{j=1}^{q} \int_{\Omega} \{ |v_j(x)|^2 + \mu_j |\nabla u_j(x)|^2 + (\mu_j + \lambda_j) |\text{div}(u(x))|^2 \} \, dx,$$

$\forall Z = ((u_j, v_j)_j) \in \mathcal{H}.$

Set $Z_j = (y_j, y_{j,t})$. The Lamé system may be recast as the first order abstract evolution equation

$$\dot{Z}_j = \mathcal{A}_j Z_j, \quad Z_j(0) = (y_j^0, y_j^1), \;\; j = 1, \; 2, \; ..., \; q,$$

where the dot denotes differentiation with respect to time,and the unbounded operator $\mathcal{A}_j$ is given by

$$\mathcal{A}_j \binom{u_j}{v_j} = \begin{pmatrix} v_j \\ \mu_j \Delta u_j + (\mu_j + \lambda_j)\nabla \mathrm{div} u_j - d \sum_{\ell=1}^{q} v_\ell \end{pmatrix}$$

with

$$\begin{aligned} D(\mathcal{A}_j) &= \Big\{ (u_j, v_j) \in [H_0^1(\Omega)]^N \times [H_0^1(\Omega)]^N; \\ &\qquad \mu_j \Delta u_j + (\mu_j + \lambda_j)\nabla \mathrm{div} u_j \in [L^2(\Omega)]^N \Big\}. \end{aligned}$$

It can be checked that one has (assuming for instance that Γ is C^2)

$$D(\mathcal{A}_j) = [H^2(\Omega) \cap H_0^1(\Omega)]^N \times [H_0^1(\Omega)]^N.$$

Thus, the operator $\mathcal{A}_j$ has a compact resolvent. Consequently the spectrum of $\mathcal{A}_j$ is discrete for each j.

With the help of Lumer-Phillips Theorem, (Pazy's book on semigroups, p. 14), one can show that the operator $\mathcal{A}$ defined by $\mathcal{A}Z = (\mathcal{A}_j Z_j)_j$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$. Indeed, $D(\mathcal{A})$ is dense in $\mathcal{H}$, $\mathcal{A}$ is dissipative

$$\Re(\mathcal{A}Z, Z) = -\int_\Omega d(x) |\sum_{j=1}^{q} v_j(x)|^2 \, dx \leq 0, \ \ \forall Z \in D(\mathcal{A}),$$

and (denoting by $\mathcal{I}$ the identity operator on $\mathcal{H}$):

$$R(\mathcal{I} - \mathcal{A}) = \mathcal{H}, \text{ by Lax-Milgram Lemma.}$$

Lamé systems with localized damping

Theorem 2: Strong stability

Let ω be a nonempty open subset of Ω. Suppose that d is positive in ω. The elastodynamic system is strongly stable:

$$\lim_{t\to\infty} E(t) = 0$$

if and only if the propagation speeds are pairwise distinct:

$$\mu_j \neq \mu_k \text{ and } \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \quad \forall j, k,\ j \neq k.$$

Proof sketch:

We may apply Dafermos criterion, or Benchimol or Arendt-Batty strong stability criterion. It suffices to show that $\mathcal{A}$ has no purely imaginary eigenvalue. One easily checks that $0 \in \rho(\mathcal{A})$. Now, let λ be a nonzero real number and let $Z = (u, v) \in D(\mathcal{A})$ with

$$\mathcal{A}Z = i\lambda Z. \qquad (*)$$

We shall show that $Z = (0, 0)$. It follows from $(*)$:

$$d(x) \sum_{j=1}^{q} u_j = 0 \text{ in } \Omega, \text{ and so, } -\lambda^2 u_j - \mu_j \Delta u_j - (\mu_j + \lambda_j) \nabla \mathrm{div} u_j = 0 \text{ in } \Omega.$$

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Therefore, setting $\varphi_j = \text{div}(u_j)$ and $\ell_j = 1/(\lambda_j + 2\mu_j)$, it follows

$$\sum_{j=1}^{q} u_j = 0 \text{ in } \omega, \text{ and } -\lambda^2 \ell_j \varphi_j - \Delta \varphi_j = 0 \text{ in } \omega.$$

Using elementary algebra, one derives from the last two equations

$$\sum_{j=1}^{q} \ell_j^k \varphi_j = 0 \text{ in } \omega, \quad k = 0,\ 1,\ ...,\ q-1.$$

The determinant of that linear system is a Vandermonde determinant and is given by

$$D_q = \prod_{1 \le j < k \le q} (\ell_k - \ell_j).$$

One checks that $D_q \neq 0$ if and only if $\lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k$ for all $j,\ k$ with $j \neq k$. In this case, $\varphi_j = 0$ in ω for each j.

$$D_q = \prod_{1 \le j < k \le q} (\ell_k - \ell_j).$$

Consequently,

$$-\lambda^2 u_j - \mu_j \Delta u_j = 0 \text{ in } \omega.$$

Repeating the same arguments as above, we find, (setting $m_j = 1/\mu_j$):

$$\sum_{j=1}^{q} m_j^k u_j = 0 \text{ in } \omega, \quad k = 0,\ 1,\ ...,\ q-1.$$

As earlier, one derives $u_j = 0$ in ω for each j if and only if $\mu_j \neq \mu_k$ for all $j,\ k$ with $j \neq k$.

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The Imanuvilov-Yamamoto Carleman estimate for the static Lamé system [Appl. Anal. 2004] then yields $u_j = 0$ in Ω for each j. Hence $Z = (0, 0)$. □

Lamé systems with localized damping

Theorem 3: Exponential stability

Let $(y_j^0, y_j^1)_j \in \left(\left[H_0^1(\Omega)\right]^N \times \left[L^2(\Omega)\right]^N\right)^q$. Suppose

$$\mu_j \neq \mu_k,\ \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k,\ \text{and } \lambda_j\mu_k = \lambda_k\mu_j, \quad \forall j, k,\ j \neq k.$$

Assume that ω is a neighborhood of $\partial\Omega$, and suppose that the damping is effective in ω:

$$\exists d_0 > 0 : d(x) \geq d_0 \text{ a.e. } \omega.$$

There exist positive constants M and κ, independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t}E(0),\ \text{for all } t \geq 0.$$

Proof Sketch:

Thanks to the semigroup exponential stability criterion of Huang or Pruss, and given that we already have strong stability, the exponential decay estimate will follow from the resolvent estimate

$$\exists C_0 > 0 : ||(ib\mathcal{I} - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C_0, \quad \forall b \in \mathbb{R}.$$

To this end, let $U \in \mathcal{H}$, and let b be a real number. Since the range of $ib\mathcal{I} - \mathcal{A}$ is $\mathcal{H}$, there exists $Z \in D(\mathcal{A})$ such that

$$ibZ - \mathcal{A}Z = U.$$

We shall prove

$$||Z||_{\mathcal{H}} \leq C_0 ||U||_{\mathcal{H}},$$

where, here and in the sequel, C_0 is a generic positive constant that may eventually depend on Ω, ω, and d, and the other parameters of the system, but not on b.

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where, here and in the sequel, C_0 is a generic positive constant that may eventually depend on Ω, ω, and d, and the other parameters of the system, but not on b.

To establish that inequality, first, we note that if $Z = (u, v)$, and $U = (f, g)$, then the equation $ibZ - \mathcal{A}Z = U$ may be recast as $(1 \leq j \leq q)$:

$$ibu_j - v_j = f_j$$

$$ibv_j - \mu_j \Delta u_j - (\lambda_j + \mu_j)\nabla \mathrm{div} u_j + d \sum_{k=1}^{q} v_j = g_j.$$

The proof of the inequality requires several steps which can be summed up as: first we shall use the multipliers technique to prove

$$||Z||_{\mathcal{H}}^2 \leq C_0||U||_{\mathcal{H}}^2 + C_0 b^2 \sum_{j=1}^{q} \int_\Omega \zeta^2 |u_j|^2 \, dx.$$

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$$||Z||^2_{\mathcal{H}} \le C_0||U||^2_{\mathcal{H}} + C_0 b^2 \sum_{j=1}^{q} \int_\Omega \zeta^2 |u_j|^2 \, dx.$$

Then using elementary algebra, we shall derive

$$||Z||^2_{\mathcal{H}} \le C_0||U||^2_{\mathcal{H}} - 2C_0 b^2 \Re \sum_{1\le j<k\le q} \int_\Omega \zeta^2 u_j \bar{u}_k \, dx.$$

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Finally, we will use appropriate multipliers and the parameter constraints to derive the claimed estimate.

Taking the inner product with Z on both sides of $ibZ - \mathcal{A}Z = U$, then taking the real parts, we immediately derive

$$\int_\Omega d(x) \left| \sum_{k=1}^{q} v_k(x) \right|^2 dx \leq ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}}.$$

It now follows from the equation $ibu_j - v_j = f_j$

$$\begin{aligned} b^2 \int_\Omega d(x) \left| \sum_{k=1}^{q} u_k(x) \right|^2 dx &\leq 2 \int_\Omega d(x) \left| \sum_{k=1}^{q} v_k(x) \right|^2 dx \\ &\quad + 2 \int_\Omega d(x) \left| \sum_{k=1}^{q} f_k(x) \right|^2 dx \\ &\leq 2||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + C_0 ||U||_{\mathcal{H}}^2. \end{aligned}$$

Let $\alpha > 0$ and β be real constants. Multiply the equation $ibv_j - \mu_j \Delta u_j - (\lambda_j + \mu_j)\nabla \mathrm{div} u_j + d\sum_{k=1}^{q} v_j = g_j$ by $\beta \bar{u}_j$, integrate on Ω, take the sum over j, and take real parts to find

$$
\begin{aligned}
&\beta \Re \int_\Omega \sum_{j=1}^{q} g_j \bar{u}_j \, dx \\
&= \beta \Re \int_\Omega (\sum_{j=1}^{q} \{ibv_j - \mu_j \Delta u_j - (\lambda_j + \mu_j)\nabla \mathrm{div} u_j + d\sum_{k=1}^{q} v_k)\bar{u}_j\} \, dx.
\end{aligned}
$$

From which, one derives (Green's formula, Cauchy Schwarz inequality, and equation $ibu_j - v_j = f_j$):

$$\beta \sum_{j=1}^{q} \left(\mu_j |\nabla u_j|_2^2 + (\lambda_j + \mu_j) |\mathrm{div} u_j|_2^2 - |v_j|_2^2 \right) \leq C_0 \left(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} \right).$$

Now, multiply the equation $ibv_j - \mu_j \Delta u_j - (\lambda_j + \mu_j) \nabla \mathrm{div} u_j + d \sum_{k=1}^{q} v_j = g_j$ by $2\alpha x \nabla \bar{u}_j$, take real parts and use Green's formula to derive:

$$
\begin{aligned}
&\sum_{j=1}^{q}\left(\alpha N|v_j|_2^2-(N-2)\alpha\left(\mu_j|\nabla u_j|_2^2+(\lambda_j+\mu_j)|\mathrm{div}u_j|_2^2\right)\right)\\
&\leq C_0\left(||U||_{\mathcal{H}}||Z||_{\mathcal{H}}+||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}}\right)\\
&\quad+C_0\sum_{j=1}^{q}\int_{\partial\Omega}\left(\mu_j|\partial_\nu u_j|^2+(\lambda_j+\mu_j)|\partial_\nu u_j\cdot\nu|^2\right)d\Gamma.
\end{aligned}
$$

Gathering those two inequalities, and choosing α and β with $\alpha(N-2) < \beta < \alpha N$, it follows:

$$\sum_{j=1}^{q} \left(|v_j|_2^2 + \left(\mu_j |\nabla u_j|_2^2 + (\lambda_j + \mu_j)|\mathrm{div} u_j|_2^2 \right) \right)$$
$$\leq C_0 \left(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} \right)$$
$$+ C_0 \sum_{j=1}^{q} \int_{\partial\Omega} (\mu_j |\partial_\nu u_j|^2 + (\lambda_j + \mu_j)|\partial_\nu u_j \cdot \nu|^2)\, d\Gamma.$$

Let ω_1 be another neighborhood of $\partial\Omega$ that is strongly contained in ω. Let $h \in \left[C^1(\bar{\Omega})\right]^N$ be a vector field with

$$h = 0 \text{ in } \Omega \setminus \omega_1, \quad h = \nu \text{ on } \partial\Omega.$$

Now, multiply the equation $ibv_j - \mu_j \Delta u_j - (\lambda_j + \mu_j)\nabla \mathrm{div} u_j + d \sum_{k=1}^{q} v_j = g_j$ by $2h\nabla \bar{u}_j$, take real parts and use Green's formula to derive:

$$
\begin{aligned}
& C_0 \sum_{j=1}^{q} \int_{\partial\Omega} (\mu_j |\partial_\nu u_j|^2 + (\lambda_j + \mu_j)|\partial_\nu u_j \cdot \nu|^2)\, d\Gamma. \\
& \leq C_0 \sum_{j=1}^{q} \int_{\omega_1} \left\{ |v_j|^2 + \left(\mu_j \nabla u_j|^2 + (\lambda_j + \mu_j)|\mathrm{div} u_j|^2 \right) \right\} dx \\
& \quad + C_0 \left(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} \right).
\end{aligned}
$$

Let $\eta \in C^1(\bar{\Omega})$ be a nonnegative function with

$$0 \leq \eta \leq 1 \text{ in } \Omega, \quad \eta = 1 \text{ in } \omega_1 \text{ and } \eta = 0 \text{ in } \Omega \setminus \omega_2,$$

where ω_2 is another neighborhood of $\partial\Omega$ with $\omega_1 \Subset \omega_2 \Subset \omega$.

Multiplying the equation $ibv_j - \mu_j \Delta u_j - (\lambda_j + \mu_j)\nabla \mathrm{div} u_j + d \sum_{k=1}^{q} v_j = g_j$

by $\eta^2 \bar{u}_j$, taking real parts and using Green's formula, we derive, (assuming $|b| > 1$):

$$C_0 \sum_{j=1}^{q} \int_{\omega_1} \left\{ |v_j|^2 + \left(\mu_j \nabla u_j|^2 + (\lambda_j + \mu_j)|\mathrm{div} u_j|^2 \right) \right\} dx$$

$$\leq C_0 b^2 \sum_{j=1}^{q} \int_{\omega_2} |u_j|^2 \, dx$$

$$+ C_0 \left(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^2 \right).$$

Now, let $\zeta \in C^1(\bar{\Omega})$ be a nonnegative function with

$$0 \leq \zeta \leq 1 \text{ in } \Omega, \quad \zeta = 1 \text{ in } \omega_2 \text{ and } \zeta = 0 \text{ in } \Omega \setminus \omega.$$

We have

$$\begin{aligned} C_0 b^2 \sum_{j=1}^{q} \int_{\omega_2} |u_j|^2 \, dx &\leq C_0 b^2 \int_\Omega \zeta^2 \left| \sum_{j=1}^{q} u_j \right|^2 dx \\ &-2C_0 b^2 \Re \sum_{1 \leq j < k \leq q} \int_\Omega \zeta^2 u_j \bar{u}_k \, dx. \end{aligned}$$

Therefore

$$\begin{aligned} b^2 \sum_{j=1}^{q} |u_j|_2^2 + ||Z||_{\mathcal{H}}^2 &\leq -2C_0 b^2 \Re \sum_{1 \leq j < k \leq q} \int_\Omega \zeta^2 u_j \bar{u}_k \, dx \\ &+C_0 \left(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^2 \right). \end{aligned}$$

Multiply the equation

$-b^2 u_j - \mu_j \Delta u_j - (\lambda_j + \mu_j)\nabla \mathrm{div} u_j + d \sum_{\ell=1}^{q} v_\ell = g_j + ibf_j$ by $\mu_k \zeta^2 \bar{u}_k$, and

the equation $-b^2 u_k - \mu_k \Delta u_k - (\lambda_k + \mu_k)\nabla \mathrm{div} u_k + d \sum_{\ell=1}^{q} v_\ell = g_k + ibf_k$

by $\mu_j \zeta^2 \bar{u}_j$, then taking real parts and using Green's formula, we find

$$\begin{aligned}
&-b^2 \mu_k \Re \int_\Omega \zeta^2 u_j \bar{u}_k \, dx + \mu_j \mu_k \Re \int_\Omega \zeta^2 \nabla u_j \nabla \bar{u}_k \, dx \\
&\quad + 2\mu_j \mu_k \Re \int_\Omega \zeta \bar{u}_k \cdot (\nabla \zeta \nabla u_j) + (\lambda_j + \mu_j)\mu_k \Re \int_\Omega \zeta^2 \mathrm{div} u_j \mathrm{div} \bar{u}_k \, dx \\
&\quad + 2(\lambda_j + \mu_j)\mu_k \Re \int_\Omega \zeta (\mathrm{div} u_j) \bar{u}_k \cdot \nabla \zeta \, dx \\
&= \mu_k \int_\Omega \zeta^2 \left(g_j + ibf_j - d \sum_{\ell=1}^{q} v_\ell \right) \bar{u}_k,
\end{aligned}$$

and

$$
\begin{aligned}
&-b^2\mu_j\Re\int_\Omega \zeta^2\bar{u}_j u_k\,dx+\mu_j\mu_k\Re\int_\Omega \zeta^2\nabla\bar{u}_j\nabla u_k\,dx\\
&\quad+2\mu_j\mu_k\Re\int_\Omega \zeta\bar{u}_j\cdot(\nabla\zeta\nabla u_k)+(\lambda_k+\mu_k)\mu_j\Re\int_\Omega \zeta^2\mathrm{div}\bar{u}_j\mathrm{div}u_k\,dx\\
&\quad+2(\lambda_k+\mu_k)\mu_j\Re\int_\Omega \zeta(\mathrm{div}u_k)\bar{u}_j\cdot\nabla\zeta\,dx\\
&=\mu_j\int_\Omega \zeta^2\left(g_k+ibf_k-d\sum_{\ell=1}^{q}v_\ell\right)\bar{u}_j.
\end{aligned}
$$

Subtracting the last equation from the preceding one, and taking the sum over the indices j and k, we derive

$$
\begin{aligned}
&-b^2 \sum_{1\le j<k\le q} \Re \int_\Omega \zeta^2 \bar{u}_j u_k \, dx \\
&= \sum_{1\le j<k\le q} \frac{2\mu_j\mu_k}{\mu_k-\mu_j} \Re \int_\Omega \zeta(\bar{u}_j \cdot (\nabla\zeta\nabla u_k) - \bar{u}_k \cdot (\nabla\zeta\nabla u_j))\, dx \\
&\quad + \sum_{1\le j<k\le q} \frac{2(\lambda_k+\mu_k)\mu_j}{\mu_k-\mu_j} \Re \int_\Omega \zeta(\bar{u}_j \mathrm{div} u_k) - \bar{u}_k \mathrm{div} u_j) \cdot \nabla\zeta \, dx \\
&\quad + \sum_{1\le j<k\le q} \frac{1}{\mu_k-\mu_j} \int_\Omega \zeta^2 (g_k + ibf_k - d\sum_{\ell=1}^{q} v_\ell)\bar{u}_j \, dx \\
&\quad - \sum_{1\le j<k\le q} \frac{1}{\mu_k-\mu_j} \int_\Omega \zeta^2 (g_j + ibf_j - d\sum_{\ell=1}^{q} v_\ell)\bar{u}_k \, dx.
\end{aligned}
$$

Cauchy-Schwarz inequality then yields

$$\left| b^2 \sum_{1\le j<k\le q} \Re \int_\Omega \zeta^2 \bar{u}_j u_k \, dx \right| \le C_0 \sum_{j=1}^{q} |u_j|_2^2$$
$$+C_0 \left(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^2 \right).$$

Hence

$$b^2 \sum_{j=1}^{q} |u_j|_2^2 + ||Z||_{\mathcal{H}}^2 \quad \le C_0 \sum_{j=1}^{q} |u_j|_2^2$$
$$+C_0 \left(||U||_{\mathcal{H}} ||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}^2 \right).$$

Choosing $|b|$ large, enough, and using Young inequality, we get the claimed inequality for $|b| > b_0$ form some $b_0 > 0$. The inequality for all real numbers b follows from the continuity of the resolvent for $|b| \leq b_0$. $\square$

Choosing $|b|$ large, enough, and using Young inequality, we get the claimed inequality for $|b| > b_0$ form some $b_0 > 0$. The inequality for all real numbers b follows from the continuity of the resolvent for $|b| \leq b_0$. □

A result of Haraux on the equivalence between observability and exponential stability (Portugal Math., 1989) shows:

An observability result

Let $T > 0$. Let ω be a neighborhood of the boundary of Ω.
Consider the uncoupled elastodynamic system

$$\begin{aligned}
&y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j)\nabla \mathrm{div}(y_j) = 0 \text{ in } \Omega \times (0, T) \\
&y_j = 0 \text{ on } \Gamma \times (0, T) \\
&y_j(x, 0) = y_j^0(x), \quad y_{jt}(x, 0) = y_j^1(x), \text{ in } \Omega, \\
&j = 1,\ 2, ...,\ q.
\end{aligned}$$

There exists $T_0 > 0$ such that for any $T > T_0$, there exists $C > 0$:

$$E(0) \leq \int_0^T \int_\omega |\sum_{j=1}^q y_{jt}(x, t)|^2 \, dxdt,$$

provided that

$$\mu_j \neq \mu_k,\ \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \text{ and } \lambda_j \mu_k = \lambda_k \mu_j, \quad \forall j, k,\ j \neq k.$$

Some references

- C. Dafermos, Wave equations with weak damping. SIAM J. Appl. Math. 18 (1970), 759-767.
- A. Haraux, On a completion problem in the theory of distributed control of wave equations. Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. X (Paris, 1987-1988), 241-271, Pitman Res. Notes Math. Ser., 220, Longman Sci. Tech., Harlow, 1991.
- J.L. Lions, *Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués*, Vol. 1, RMA 8, Masson, Paris, 1988.
- H. E. Wilhelm, T. J. van der Werff, Nonlinear wave equations for chemical reactions with diffusion in multicomponent systems. J. Chem. Phys. 67 (1977), 3382-3387.

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!