Simultaneous and indirect control of waves: some recent developments and open problems

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Identification and Control: Some challenges
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Overview

- Indirect Controllability

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  - Brief literature

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  - Hyperbolic equations with internal coupling.

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- Indirect Stabilization

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  - Mindlin-Timoshenko plate
  - Kirchhoff plate-wave

Notations

Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
Γ= boundary of Ω is smooth,
$T > 0$, $Q = \Omega \times (0, T)$
$\omega =$ nonvoid open subset in Ω.
The coefficients matrix $(b_{ij})_{i,j}$, satisfies:

$$b_{ij} \in C^1(\bar{\Omega}); \quad b_{ij} = b_{ji}, \quad \forall i,\ j = 1, 2, ..., N,$$
$$\exists a_0 > 0 : b_{ij}(x) z_i z_j \geq a_0 z_i z_i, \quad \forall (x, z) \in \bar{\Omega} \times \mathbb{R}^N.$$

The Einstein summation convention on repeated indices is used throughout.
a, b, c, d lie in $L^\infty(0, T; L^s(\Omega))$, $s \geq \max(2, N)$ for $N \neq 2$,
and $s > 2$ for $N = 2$.
k_{ij}, l_{ij} lie in $W_0^{1,\infty}(0, T; L^s(\Omega))$.

Controllability

Consider the controllability problems: Given (z^0, z^1) and (w^0, w^1), and $\varepsilon > 0$, find a control h such that if (z, w) solves the system

$$
\begin{cases}
z_{tt} - \partial_i(b_{ij}(x)\partial_j z) + az + cw - \mathrm{div}(k_{11}z) - (l_{11}z)_t \\
-\mathrm{div}(k_{21}w) - (l_{21}w)_t = h1_\omega \text{ in } Q \\
\\
w_{tt} - \partial_i(b_{ij}(x)\partial_j w) + bz + dw - \mathrm{div}(k_{12}z) - (l_{12}z)_t \\
-\mathrm{div}(k_{22}w) - (l_{22}w)_t = 0 \text{ in } Q \\
\\
z = 0, \quad w = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\
\\
z(0) = z^0; \quad z_t(0) = z^1 \quad w(0) = w^0; \quad w_t(0) = w^1 \text{ in } \Omega,
\end{cases}
$$

then (exact controllability)

$$z(T) = 0, \quad z_t(T) = 0, \quad w(T) = 0, \quad w_t(T) = 0 \text{ in } \Omega,$$

or else (approximate controllability)

$$||z(T)||_1 + ||z_t(T)||_2 \leq \varepsilon, \quad ||w(T)||_1 + ||w_t(T)||_2 \leq \varepsilon.$$

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- For approximate controllability, only T must be large enough.
- Lions' HUM reduces exact controllability to an inverse (observability) estimate for the adjoint system.

Brief literature

- Dáger (2006), $\Omega = (0, 1)$, $T \geq 4$, $b = -1_{\mathcal{O}}$, all other *l.o.t* vanish. Proved a weaker estimate; see Theorem 2 in the sequel.

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- Rosier-de Teresa (2011), $\Omega = (0, 1)$, $T \geq 4$, $b = -a(x)^2$, $a \in L^\infty(\Omega)$, all other *l.o.t* vanish.
- Alabau-Leautaud (2012), $c = b$, $d = a$ are smooth enough, and $||b||_\infty$ is small, all other *l.o.t* vanish, ω and $\mathcal{O}$ may have empty intersection, and both satisfy
  (GCC) [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω, (resp. $\mathcal{O}$) in a time less than T.
  But the controllability time blows up as the norm of the coupling function b goes to zero; this is not natural. One would expect the controllability cost to blow up as the coupling goes to zero, but not the controllability time.

Observability estimates

Consider the coupled (adjoint) system

$$\begin{cases} u_{tt} - \partial_i(b_{ij}(x)\partial_j u) + au + bv + k_{11} \cdot \nabla u + l_{11} u_t \\ +k_{12} \cdot \nabla v + l_{12} v_t = 0 \text{ in } Q \\ \\ v_{tt} - \partial_i(b_{ij}(x)\partial_j v) + cu + dv + k_{21} \cdot \nabla u + l_{21} u_t \\ +k_{22} \cdot \nabla v + l_{22} v_t = 0 \text{ in } Q \\ \\ u = 0, \quad v = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ \\ u(0) = u^0; \quad u_t(0) = u^1 \quad v(0) = v^0; \quad v_t(0) = v^1 \text{ in } \Omega. \end{cases}$$

The coupled system is well-posed in $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$.

Introduce the energies:

$$E_u(t) = \frac{1}{2}\int_\Omega \{|u_t(x,t)|^2 + (b_{ij}(x)\partial_j u(x,t)\partial_i u(x,t)\}\, dx,$$

$$\widehat{E}_u(t) = \frac{1}{2}\left(||u(.,t)||^2_{L^2(\Omega)} + ||u_t(.,t)||^2_{H^{-1}(\Omega)}\right).$$

For each $t \in [0, T]$, set

$$E(t) = E_u(t) + \widehat{E}_v(t), \quad \widehat{E}(t) = \widehat{E}_u(t) + \widehat{E}_v(t).$$

Introduce a function $q \in C^2(\bar{\Omega})$ satisfying for some $m_0 \geq 4$:

i) $\left(2b_{il}(b_{kj}q_{x_k})_{x_l} - b_{ij,x_l}b_{kl}q_{x_k}\right) z_i z_j \geq m_0 b_{ij} z_i z_j, \quad \forall (x,z) \in \bar{\Omega} \times \mathbb{R}^N.$
ii) $\min\left\{|\nabla q(x)|; x \in \bar{\Omega}\right\} > 0.$
iii) $\frac{1}{4} b_{ij}(x) q_{x_i}(x) q_{x_j}(x) \geq R_1^2 \geq R_0^2 > 0, \quad \forall x \in \bar{\Omega},$

where $R_0 = \min\left\{\sqrt{q(x)}; x \in \bar{\Omega}\right\}$, and $R_1 = \max\left\{\sqrt{q(x)}; x \in \bar{\Omega}\right\}$. Let ν be the unit normal pointing into the exterior of Ω, and set

$$\Gamma_0 = \left\{x \in \partial\Omega; b_{ij}\nu_i q_{x_j}(x) > 0\right\}.$$

Theorem 1 (AMOP 2012)

Let ω and $\mathcal{O}$ be neighborhoods of Γ_0. Let $r \in \mathcal{D}(0, T)$ be an appropriate cutoff function. Assume that $a,\ c,\ d \in L^\infty(0, T; L^s(\Omega))$, with $s > 2$ for $N \in \{1,\ 2\}$ and $s \geq N$ for $N \geq 3$. Let $b \in L^\infty(Q)$, and let $k_{ij} \in (W_0^{1,s}(Q) \cap L^\infty(Q))^N$, $l_{ij} \in W_0^{1,s}(Q) \cap L^\infty(Q)$, $i, j = 1, 2$. Suppose that $k_{12} \equiv 0$, $l_{12} \equiv 0$, $\mathrm{supp}(k_{22}) \subset \omega_0 \times (0, T)$, and $\mathrm{supp}(l_{22}) \subset \omega_0 \times (0, T)$, where ω_0 is another neighborhood of Γ_0 whose closure $\bar{\omega}_0$ is contained in $\mathcal{O} \cap \omega$. Suppose that there exists $b_0 > 0$ such that $b(x, t) \geq b_0$ for almost every (x, t) in $\mathcal{O} \times (0, T)$.

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For every $T > 2R_1$, there exists a positive constant C such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $(v^0, v^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, one has the observability estimate:

$$E(0) \leq C \int_0^T \int_\omega (r^2 |u_t|^2 + |u|^2)\, dxdt$$

for the corresponding solution pair (u, v) of the adjoint system.

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4. The support constraints on k_{22} and l_{22} are used in the proof of the observability estimate to absorb some unwanted terms, but they may be replaced with smallness constraints instead.

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3. The restrictions $k_{12} \equiv 0$ and $l_{12} \equiv 0$ are for well-posedness purposes.
4. The support constraints on k_{22} and l_{22} are used in the proof of the observability estimate to absorb some unwanted terms, but they may be replaced with smallness constraints instead.
5. Once Theorem 1 is proven, the Hilbert uniqueness method (H.U.M) of Lions may be used to show that the optimal control for the controllability problem is given by $h = (r^2 \hat{u}_t)_t - \hat{u}$ where $(\hat{u}, \hat{v})$ denotes an appropriate solution of the adjoint system.

Proof of Theorem 1: key elements

- Energy estimates show

$$E(0) \leq C \int_{Q_0} \{|u_t|^2 + |\nabla u|^2 + |v|^2\}\, dxdt,$$

where Q_0 is an appropriate subset of Q.

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- Fu-Yong-Zhang Carleman estimate shows

$$\int_{Q_0} (|u_t|^2 + |\nabla u|^2 + |v|^2)\, dxdt \leq Ce^{-\mu\lambda} E(0) + C \int_0^T r^2 \int_{\omega_0} |v|^2\, dxdt$$
$$+ C \int_0^T \int_\omega (r^2 |u_t|^2 + |u|^2)\, dxdt$$

  where $\lambda > 0$ is large enough, and $\mu > 0$ is fixed.

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where $\lambda > 0$ is large enough, and $\mu > 0$ is fixed.

- Use a localizing argument to absorb $C \int_0^T r^2 \int_{\omega_0} |v|^2 \, dxdt$.

A Carleman estimate for two coupled systems

Lemma 1

Let b_{ij} be given as above. Assume that Γ_0 as in Theorem 1, and ω is any neighborhood of Γ_0. Then there exists $\lambda_0 > 1$ and a positive constant $C = C(\Omega, T)$, such that for all $a,\ b,\ c,\ d \in L^\infty(0, T; L^s(\Omega))$, with $s > 2$ for $N = 2$ and $s \geq \max(2, N)$ for $N \neq 2$, all $\lambda \geq \lambda_0$ and all $u,\ v \in C([0, T]; L^2(\Omega))$ satisfying
$u(x, 0) = u(x, T) = v(x, 0) = v(x, T) = 0$ for $x \in \Omega$, $\mathcal{P}u \in H^{-1}(Q)$, $\mathcal{P}v \in H^{-1}(Q)$, and $(u, \mathcal{P}\eta) = \langle \mathcal{P}u, \eta\rangle_{H^{-1}(Q), H_0^1(Q)}$,
$(v, \mathcal{P}\eta) = \langle \mathcal{P}v, \eta\rangle_{H^{-1}(Q), H_0^1(Q)}, \quad \forall \eta \in H_0^1(Q)$ with $\mathcal{P}\eta \in L^2(Q)$,one has:

A Carleman estimate for two coupled systems

$$
\begin{aligned}
&\lambda \int_Q e^{2\lambda\varphi}(u^2+v^2)\,dxdt \\
&\le C\left(||e^{\lambda\varphi}(\mathcal{P}u+au+bv)||^2_{H^{-1}(Q)} + ||e^{\lambda\varphi}(\mathcal{P}v+cu+dv)||^2_{H^{-1}(Q)}\right) \\
&\quad + \frac{C}{\lambda^{(2-2N/s)}}\Big(||e^{\lambda\varphi}(au+bv)||^2_{L^2(0,T;H^{-N/s}(\Omega))} \\
&\qquad\qquad + ||e^{\lambda\varphi}(cu+dv)||^2_{L^2(0,T;H^{-N/s}(\Omega))}\Big) \\
&\quad + C\lambda^2||e^{\lambda\varphi}u||^2_{L^2(0,T;L^2(\omega))} + C\lambda^2||e^{\lambda\varphi}v||^2_{L^2(0,T;L^2(\omega))}.
\end{aligned}
$$

Set

$$\delta = ||a||_{\infty,s} + ||b||_{\infty,s} + ||c||_{\infty,s} + ||d||_{\infty,s} + \sum_{i,j=1}^{2} ||\mathrm{div}(k_{ij})||_{\infty,s} + \sum_{i,j=1}^{2} ||l_{ij,t}||_{\infty,s}$$

$$\delta_0 = \sum_{i,j=1}^{2} ||k_{ij}||_{\infty} + \sum_{i,j=1}^{2} ||l_{ij}||_{\infty}$$

where $||.||_{\infty,s} = ||.||_{L^\infty(0,T;L^s(\Omega))}$, and $||.||_{\infty} = ||.||_{L^\infty(Q)}$.

Theorem 2. (AMOP 2012)

Let ω, $\mathcal{O}$, a, d and s be as in Theorem 1, and suppose that $b \in L^\infty(0, T; L^s(\Omega))$, $c \in L^\infty(Q)$, and there exists $b_0 > 0$ such that $b(x, t) \geq b_0$ for almost every (x, t) in $\mathcal{O} \times (0, T)$. Let $k_{ij} \in (W_0^{1,s}(Q) \cap L^\infty(Q))^N$, $l_{ij} \in W_0^{1,s}(Q) \cap L^\infty(Q)$, $i, j = 1, 2$. Suppose that $k_{21} \equiv 0$, $l_{21} \equiv 0$, $\text{supp}(k_{ij}) \subset \omega_0 \times (0, T)$, and $\text{supp}(l_{ij}) \subset \omega_0 \times (0, T)$.

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For every $T > 2R_1$, there exists a positive constant $C_0 = C_0(\Omega, \omega, \mathcal{O}, T, N, s)$ such that for all $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, one has the observability estimate:

$$\widehat{E}(0)^2 \leq e^{C_0(1+\delta_0+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |u|^2 \, dxdt \right) (\widehat{E}_u(0) + E_v(0))$$

for all solution pair (u, v) of the adjoint system.

Proof of Theorem 2: Main ideas

Step 1. Prove the energy estimates

- $$\widehat{E}(t) \leq \left[\exp C_0(1+\delta_0+\delta^{\frac{N+s}{2s}})|t-\tau|\right] \widehat{E}(\tau), \qquad \forall \tau, t \in [0, T],$$

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- $$\int_{T_0}^{T_0'} h\widehat{E}(t)\,dt \le C_0(1+\delta+\delta_0)\int_{Q_0}\{|u|^2+|v|^2\}\,dxdt,$$

where h is an appropriate cut-off function.

Step 2. Derive from Step 1

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Step 3. Duyckaerts-Zhang-Zuazua Carleman estimate yields

$$\int_{Q_0} (|u|^2 + |v|^2)\, dxdt \leq e^{-C_0\lambda}\widehat{E}(0) + e^{C_0\lambda} \int_0^T \int_\omega |u|^2\, dxdt + e^{C_0\lambda} \int_0^T r^2 \int_{\omega_0} |v|^2\, dxdt,$$

for some constants $C_0 = C_0(\Omega, T, N, s, \omega) > 0$, and for all $\lambda \geq C_0(1 + \delta_0 + \delta^{\frac{2s}{3s-2N}})$.

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Step 4. Use a localizing argument to absorb $e^{C_0\lambda} \int_0^T r^2 \int_{\omega_0} |v|^2 \, dxdt$.

Let $a,\ b,\ c,\ d \in L^s(\Omega)$, with s as in Theorem 1. Assume now $l_{ij} \equiv 0$, and $k_{ij} \equiv 0$, $i, j = 1, 2$. Let ω, $\mathcal{O}$, be as in Theorem 1, and suppose that there exists $b_0 > 0$ such that $b(x) \geq b_0$ for almost every x in $\mathcal{O}$.
Further assume that either:

$$a \geq 0, \quad d \geq 0, \quad 2a - |b + c| \geq 0, \text{ and } 2d - |b + c| \geq 0, \text{ a.e. } x \in \Omega$$

or else

$$a \geq 0, \quad d \geq 0, \text{ a.e. } x \in \Omega, \quad 1 - C_s^2 |b + c|_s > 0, \text{ and } \lambda_0^2 - |b + c|_s > 0,$$

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where λ_0^2 is the first eigenvalue of the operator $-\partial_i (b_{ij}(x) \partial_j)$ under Dirichlet boundary conditions, and C_s denotes the best constant in the Sobolev inequality:

$$||w||^2_{\frac{2s}{s-2}} \leq C_s^2 \int_\Omega b_{ij}(x) \partial_j w(x) \partial_i w(x)\, dx, \quad \forall w \in H_0^1(\Omega).$$

Theorem 3

Assume the hypotheses just stated. For every $T > 2R_1$, there exists a positive constant $C_0 = C_0(\Omega, \omega, \mathcal{O}, T, N, s)$ such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $(v^0, v^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, one has the observability estimate:

$$(E_u(0) + E_v(0))^2 \leq e^{C_0(1+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |u_t|^2 \, dxdt \right) (E_u(0) + \check{E}_v(0))$$

for all solution pair (u, v) of the adjoint system, and where $2\check{E}_v(0) = ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H_0^1(\Omega)}$.

Sketch of the proof of Theorem 3

For this proof, we shall use Theorem 2, and the following result

Lemma 2

Let $a,\ b,\ c$, and d be given as in Theorem 3. Then there exists a positive constant $C_0 = C_0(\Omega, b+c)$ such that

$$||-\partial_i(b_{ij}(x)\partial_j u) + au + bv||^2_{H^{-1}(\Omega)} + ||-\partial_i(b_{ij}(x)\partial_j v) + cu + dv||^2_{H^{-1}(\Omega)}$$
$$\geq C_0 \int_\Omega \{b_{ij}(x)\partial_j u \partial_i u + b_{ij}(x)\partial_j v \partial_i v\}\, dx, \quad \forall u, v \in H^1_0(\Omega).$$

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$$\geq C_0 \int_\Omega \{b_{ij}(x)\partial_j u \partial_i u + b_{ij}(x)\partial_j v \partial_i v\}\, dx, \quad \forall u, v \in H^1_0(\Omega).$$

Set $\hat{w} = u_t$ and $\hat{z} = v_t$. Then these functions solve the system

$$\begin{cases} \hat{w}_{tt} - \partial_i(b_{ij}(x)\partial_j \hat{w}) + a\hat{w} + b\hat{z} = 0 \text{ in } Q \\ \hat{z}_{tt} - \partial_i(b_{ij}(x)\partial_j \hat{z}) + c\hat{w} + d\hat{z} = 0 \text{ in } Q \\ \hat{w} = 0, \quad \hat{z} = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ \hat{w}(0) = u^1 \in L^2(\Omega); \quad \hat{w}_t(0) = \partial_i(b_{ij}(x)\partial_j u^0) - au^0 - bv^0 \in H^{-1}(\Omega) \\ \hat{z}(0) = v^1 \in H^1_0(\Omega); \quad \hat{z}_t(0) = \partial_i(b_{ij}(x)\partial_j v^0) - cu^0 - dv^0 \in L^2(\Omega). \end{cases}$$

Introduce the following energy associated with that system

$$\widehat{E}_{\hat{w},\hat{z}}(t) = \widehat{E}_{\hat{w}}(t) + \widehat{E}_{\hat{z}}(t) \quad \forall t \in [0, T].$$

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Thanks to Theorem 2, one has:

$$\widehat{E}_{\hat{w},\hat{z}}(0)^2 \leq e^{C_0(1+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |\hat{w}|^2 \, dxdt \right) (\widehat{E}_{\hat{w}}(0) + E_{\hat{z}}(0)).$$

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Hence

$$(E_u(0) + E_v(0))^2 \leq e^{C_0(1+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |u_t|^2 \, dxdt \right) (E_u(0) + \check{E}_v(0)).$$

Theorem 4

Suppose that the hypotheses of Theorem 3 hold. For every $T > 2R_1$, there exists a positive constant $C = C(\Omega, \omega, \mathcal{O}, T, N, s, a, b, c, d)$ such that for all $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, and $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, one has the observability estimate:

$$\check{E}_u(0) + E_v(0) \leq C \int_0^T \int_\omega \{|u_t|^2 + |u_{tt}|^2\} \, dxdt.$$

Mindlin-Timoshenko plate

$$\rho_1 y_{tt} - k\mathrm{div}(\nabla y + z) = 0 \text{ in } \Omega \times (0,\infty)$$
$$\rho_2 z_{tt} - \mu\Delta z - (\lambda+\mu)\nabla\mathrm{div} z + k(\nabla y + z) + a z_t = 0 \text{ in } \Omega \times (0,\infty)$$
$$y = 0, \quad z = 0 \text{ on } \partial\Omega \times (0,\infty)$$
$$y(.,0) = y^0 \in H_0^1(\Omega), \quad y_t(.,0) = y^1 \in L^2(\Omega),$$
$$z(.,0) = z^0 \in [H_0^1(\Omega)]^N, \quad z_t(.,0) = z^1 \in [L^2(\Omega)]^N.$$

Mindlin-Timoshenko plate

$$
\begin{aligned}
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\end{aligned}
$$

In the one-dimensional setting, the system , known as the Timoshenko beam equations, describes the motion of a beam when the effects of rotatory inertia are accounted for; the transverse displacement is represented by y while z denotes the shear angle displacement.

In 2D, that system is known as the Mindlin-Timoshenko plate equations, where y represents the vertical deflection and z stands for the rotation angles of a filament.

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The constants ρ_1, ρ_2, *k*, and μ are physical constants and are all positive. In particular, the constants λ and μ are the Lamé constants with $\lambda + \mu > 0$.

Mindlin-Timoshenko plate

2009: Fernández-Sare shows that the system is polynomially stable.

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$$(*) \qquad \frac{k}{\rho_1} = \frac{2\mu + \lambda}{\rho_2}.$$

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Questions: Is the Mindlin-Timoshenko system exponentially stable under $(*)$? What happens when $(*)$ fails?

To study the stabilization problem at hand, we are going to recast the plate system as an abstract evolution system. To this end, setting $Z = \begin{pmatrix} y \\ y' \\ z \\ z' \end{pmatrix}$, the Mindlin-Timoshenko system may then be recast as:

$$Z' - \mathcal{A}Z = 0 \text{ in } (0, \infty), \qquad Z(0) = \begin{pmatrix} y^0 \\ y^1 \\ z^0 \\ z^1 \end{pmatrix}$$

where the unbounded operator $\mathcal{A}$ is given by

Mindlin-Timoshenko plate

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{k}{\rho_1}\Delta & 0 & \frac{k}{\rho_1}\mathrm{div} & 0 \\ 0 & 0 & 0 & I \\ -\frac{k}{\rho_2}\nabla & 0 & \frac{\mu}{\rho_2}\Delta + \frac{\lambda+\mu}{\rho_2}\nabla\mathrm{div} - \frac{k}{\rho_2}I & -\frac{a}{\rho_2}I \end{pmatrix}$$

with,

$$D(\mathcal{A}) = \Big\{(u,v,w,z) \in \left(H_0^1(\Omega)\right)^2 \times \left([H_0^1(\Omega)]^N\right)^2; \\ k\mathrm{div}(\nabla u + w) \in L^2(\Omega), \\ \text{and } \mu\Delta w + (\lambda+\mu)\nabla\mathrm{div} w - k(\nabla u + w) - az \in [L^2(\Omega)]^N\Big\}$$

It can be checked that one has (assuming for instance that Γ is C^2)

$$D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times ([H^2(\Omega)]^N \cap [H_0^1(\Omega)]^N) \times [H_0^1(\Omega)]^N.$$

Mindlin-Timoshenko plate

Thus, the operator $\mathcal{A}$ has a compact resolvent. Consequently the spectrum of $\mathcal{A}$ is discrete.

Introduce the Hilbert space over the field $\mathbb{C}$ of complex numbers $\mathcal{H} = H_0^1(\Omega) \times L^2() \times [H_0^1(\Omega)]^N \times [L^2(\Omega)]^N$, equipped with the norm

$$||Z||_{\mathcal{H}}^2 = \rho_1 |v|_2^2 + k|\nabla u + w|_2^2 + \rho_2 |z|_2^2 + \mu |\nabla w|_2^2 + (\lambda + \mu)|\mathrm{div} w|_2^2,$$

$$\forall Z = (u, v, w, z) \in \mathcal{H}.$$

Mindlin-Timoshenko plate

Theorem 5: Strong stability

Suppose that ω is an arbitrary nonempty open set in Ω. Let the damping coefficient a be positive in ω. The operator $\mathcal{A}$ generates a C_0 semigroup of contractions $(S(t))_{t\geq 0}$ on the Hilbert space $\mathcal{H}$, which is strongly stable:

$$\lim_{t\to\infty} ||S(t)Z^0||_{\mathcal{H}} = 0, \quad \forall Z^0 \in \mathcal{H}.$$

Proof of Theorem 5

First, we shall prove that the unbounded operator $\mathcal{A}$ generates a C_0 semigroup of contractions $(S(t))_{t\geq 0}$, then we shall show that $i\mathbb{R} \subset \rho(\mathcal{A})$.
We have:

- $\overline{D(\mathcal{A})} = \mathcal{H}$ and the operator $\mathcal{A}$ is dissipative, as:

$$\Re\left(\mathcal{A}Z, Z\right) = -\int_\Omega a(x)|z(x)|^2\, dx \leq 0, \quad \forall Z = (u, v, w, z) \in \mathcal{D}(\mathcal{A}).$$

- $\mathcal{I} - \mathcal{A}$ is onto, by Lax-Milgram Lemma, ($\mathcal{I}$ denotes the identity operator).

To prove strong stability, it suffices, thanks to a Benchimol strong stability criterion, for linear operators with compact resolvent, to show that $\mathcal{A}$ has no imaginary eigenvalue. One easily checks that zero is not an eigenvalue of $\mathcal{A}$. Now, let b be a nonzero real number, and let $Z = (u, v, w, z) \in D(\mathcal{A})$ such that $\mathcal{A}Z = ibZ$. We shall prove that $Z = (0, 0, 0, 0)$. Note that $\mathcal{A}Z = ibZ$ may be recast as:

$$
\begin{aligned}
&-b^2 u - \hat{k}\,\mathrm{div}(\nabla u + w) = 0 \text{ in } \Omega \\
&-b^2 w - \hat{\mu}\Delta w - (\hat{\mu} + \hat{\ })\nabla \mathrm{div} w + \check{k}(\nabla u + w) + ib\check{a}w = 0 \text{ in } \Omega.
\end{aligned}
$$

One easily checks that $w = 0$ in ω, by multiplying the first equation by $\bar{u}$, the second by $\bar{w}$, integrating by parts and taking the imaginary parts. Using the second equation, it follows that $\nabla u = 0$ in ω; so, using the first equation, one derives that $u = 0$ in ω as b is nonzero.

Now, thanks to a Carleman estimate of imanuvilov-Puel for elliptic equations, there exist positive constants C, $\tau_0 \geq 1$ and $s_0 \geq 1$ such that, for every $\tau \geq \tau_0$ and every $s \geq s_0$, the component u satisfies:

$$
\begin{aligned}
& s\tau^2 \int_\Omega \{|\nabla u|^2 + s^2\tau^2\varphi^2|u|^2\} e^{2s\varphi}\, dx \\
& \leq C \int_\Omega \{b^4|u|^2 + |\mathrm{div} w|^2\} e^{2s\varphi}\, dx \\
& \quad + Cs\tau^2 \int_\omega \{|\nabla u|^2 + s^2\tau^2\varphi^2|u|^2\} e^{2s\varphi}\, dx, \quad \forall b \in \mathbb{R},
\end{aligned}
$$

where φ is an appropriate weight function, and C is independent of s, τ, and b.

Similarly, according a Carleman estimate of in Imanuvilov-Yamamoto for the static Lamé system, there exist positive constants C, $\tau_1 \geq 1$ and $s_1 \geq 1$ such that, for every $\tau \geq \tau_1$ and every $s \geq s_1$, the component w satisfies:

$$
\begin{aligned}
&\tau^2 \int_\Omega \{|\nabla w|^2 + s^2\tau^2\varphi^2|w|^2\} e^{2s\varphi}\, dx \\
&\leq C \int_\Omega \{(b^4+1)|w|^2 + |\nabla u|^2\} e^{2s\varphi}\, dx \\
&\quad + C\tau^2 \int_\omega \{|\nabla w|^2 + s^2\tau^2\varphi^2|w|^2\} e^{2s\varphi}\, dx, \quad \forall b \in \mathbb{R},
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&\tau^2 \int_\Omega \{|\nabla w|^2 + s^2\tau^2\varphi^2|w|^2\} e^{2s\varphi}\, dx \\
&\leq C \int_\Omega \{(b^4+1)|w|^2 + |\nabla u|^2\} e^{2s\varphi}\, dx \\
&\quad + C\tau^2 \int_\omega \{|\nabla w|^2 + s^2\tau^2\varphi^2|w|^2\} e^{2s\varphi}\, dx, \quad \forall b \in \mathbb{R},
\end{aligned}$$

where φ is an appropriate weight function, and C is independent of s, τ and b.

Combining those two estimates, noticing that $|\mathrm{div} w(x)| \leq |\nabla w(x)|$ for almost every x in Ω, and choosing s and τ large enough, one can use the left hand side to absorb the first integrals in the right hand side. Next, using the fact that $u = 0$ and $w = 0$ in , one derives that $u = 0$ and $w = 0$ in ; hence $Z = (0, 0, 0, 0)$. $\square$

Theorem 6: exponential stability

Suppose that the damping coefficient a satisfies:

$$\exists a_0 > 0 : a(x) \geq a_0, \quad \text{a.e. } x \in \Omega.$$

Assume that $\frac{k}{\rho_1} = \frac{2\mu+\lambda}{\rho_2}$. The semigroup $(S(t))_{t\geq 0}$ is exponentially stable, viz., there exist positive constants M and ζ with:

$$||S(t)Z^0||_{\mathcal{H}} \leq M \exp(-\zeta t)||Z^0||_{\mathcal{H}}, \quad \forall Z^0 \in \mathcal{H}.$$

Proof of Theorem 6:Main ideas

We shall use the frequency domain approach, which amounts to showing the two facts :

1. $i\mathbb{R} \subset \rho(\mathcal{A})$, and
2. $\sup\{||(ib - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})};\ b \in \mathbb{R}\} < \infty$.

Thanks to the proof of Theorem 5, we already have the first point. It remains to prove the second point. For this purpose, it suffices to show that there exists $C_0 > 0$ such that for every $U \in \mathcal{H}$, one has:

$$||(ib - \mathcal{A})^{-1}U||_{\mathcal{H}} \leq C_0||U||_{\mathcal{H}}, \quad \forall b \in \mathbb{R},$$

where hereafter, C_0 denotes a generic positive constant that may eventually depend on Ω, ω, and the other parameters of the system, but never on b.

Let $b \in \mathbb{R}$, $U = (f, g, h, l) \in \mathcal{H}$, and let $Z = (u, v, w, z) \in D(\mathcal{A})$ such that

$$(ib - \mathcal{A})Z = U.$$

We shall prove $||Z||_{\mathcal{H}} \leq C_0 ||U||_{\mathcal{H}}$. To this end, multiply both sides of the equation by Z, then take the real part of the inner product in $\mathcal{H}$ to derive :

$$\int_\Omega a(x)|z(x)|^2\,dx = \Re(U, Z) \leq ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}, \text{ or } |z|_2^2 \leq C_0||U||_{\mathcal{H}}||Z||_{\mathcal{H}}.$$

Equation $(ib - \mathcal{A})Z = U$ may be recast as:

$$\begin{cases} ibu - v = f \\ ibv - \hat{k}\mathrm{div}(\nabla u + w) = g \\ ibw - z = h \\ ibz - \hat{\mu}\Delta w - (\hat{\lambda} + \hat{\mu})\nabla \mathrm{div} w + \check{k}(\nabla u + w) + \check{a}z = l. \end{cases}$$

Thanks to the estimate on z, we derive from the third equation

$$b^2 \int_\Omega |w(x)|^2\,dx \leq C_0||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + C_0||U||_{\mathcal{H}}^2.$$

Multiplying the last equation in by $\bar{w}$, taking the real parts, and integrating by parts over Ω, one finds

$$
\begin{aligned}
&\Re ib \int_\Omega z \cdot \bar{w}\, dx + \int_\Omega \{\hat{\mu}|\nabla w|^2 + (\hat{\lambda} + \hat{\mu})|\mathrm{div} w|^2\}\, dx \\
&= \Re \int_\Omega \{(l - \check{a}z) \cdot \bar{w}\, dx - \check{k}\Re \int_\Omega (\nabla u + w) \cdot \bar{w}\, dx.
\end{aligned}
$$

With Cauchy-Schwarz inequality, one readily derives, for every b with $|b| > 1$:

$$
\begin{aligned}
&|z|_2^2 + \int_\Omega \{\hat{\mu}|\nabla w|^2 + (\hat{\lambda} + \hat{\mu})|\mathrm{div} w|^2\}\, dx \\
&\leq C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + |b|^{-1}||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}})
\end{aligned}
$$

At this stage, we note that the last inequality provides a good estimate for w; so, in order to complete the proof of Theorem 6, it remains to get estimates for u and v. This is the crucial point in the proof. First, we will estimate $|v|_2$ in terms of $|\nabla u + w|_2$, then we will estimate $|\nabla u + w|_2$. In particular, the factor $|b|^{-1}$ will be very helpful in the proof of the polynomial decay estimate given by the next theorem, leading to a better decay estimate; it can be dropped in the rest of the proof of Theorem 6 for $|b| > 1$.

Estimating $|v|_2$. Taking the conjugate of the equation $ibu - v = f$, then multiplying the new equation by $-v$ and integrating over Ω, we find

$$ib\int_\Omega \bar{u}v\,dx + |v|_2^2 = -\int_\Omega \bar{f}v\,dx.$$

Next, multiplying the equation $ibv - \hat{k}\text{div}(\nabla u + w) = g$ by $-\bar{u}$ and using Green's formula, we find

$$\begin{aligned} -ib\int_\Omega \bar{u}v\,dx &= \hat{k}\int_\Omega (\nabla u + w)\cdot\nabla\bar{u} - \int_\Omega \bar{u}g\,dx \\ &= \hat{k}\int_\Omega |\nabla u + w|^2\,dx - \hat{k}\int_\Omega (\nabla u + w)\cdot\bar{w}\,dx - \int_\Omega \bar{u}g\,dx. \end{aligned}$$

Now, thanks to Cauchy-Schwarz and Poincaré inequalities, one gets

$$\left|\int_\Omega (\bar{f}v + g\bar{u})\,dx\right| \leq |f|_2|v|_2 + |g|_2|u|_2 \leq C_0||U||_{\mathcal{H}}||Z||_{\mathcal{H}}.$$

It then follows

$$|v|_2^2 \leq C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}}) + \hat{k}|\nabla u + w|_2^2.$$

Using the speeds equality constraint, one finds:

$$\check{k} \int_\Omega |\nabla u + w|^2 \, dx \leq C_0(||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}}) + \hat{k}\Re \int_\Gamma \varphi \partial_\nu \bar{u} \, d\gamma.$$

Using appropriate first order multipliers, one gets rid of the boundary integral, and the claimed estimate follows. Applying the continuity of the resolvent and Huang or Pruss exponential stability criterion, one obtains the desired exponential stability of the semigroup. □

Theorem 7: Polynomial stability

Suppose that the damping coefficient a is as in Theorem 6. Assume that $\frac{k}{\rho_1} \neq \frac{2\mu+\lambda}{\rho_2}$. The semigroup $(S(t))_{t\geq 0}$ is polynomially stable, `viz.`, there exists a positive constant M such that:

$$||S(t)Z^0||_{\mathcal{H}} \leq M\frac{||Z^0||_{D(\mathcal{A})}}{(1+t)^{\frac{1}{2}}}, \quad \forall Z^0 \in D(\mathcal{A}).$$

Kirchhoff plate-wave

Joint work with Ahmed Hajej (U. Cergy-Pontoise, France) and Zayd Hajjej (U. Gabes, Tunisia), JMAA 2019

Undamped Kirchhoff plate/ damped wave

Consider the following weakly coupled system of Kirchhoff plate and wave equations:

$$
\begin{cases}
u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + \alpha v = 0 & \text{in} \quad \Omega \times (0, \infty) \\
v_{tt} - \Delta v + v_t + \alpha u = 0 & \text{in} \quad \Omega \times (0, \infty) \\
u = \partial_\nu u = 0 & \text{on} \quad \Gamma_0 \times (0, \infty) \\
\Delta u + (1 - \mu) B_1 u = 0 & \text{on} \quad \Gamma_1 \times (0, \infty) \\
\partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1 - \mu) B_2 u = 0 & \text{on} \quad \Gamma_1 \times (0, \infty) \\
v = 0 & \text{on} \quad \Gamma \times (0, \infty) \\
u(0) = u^0 \in V, \;\; u_t(0) = u^1 \in H_0^1(\Omega), & \\
v(0) = v^0 \in H_0^1(\Omega), \;\; v_t(0) = v^1 \in L^2(\Omega). &
\end{cases}
$$

Undamped Kirchhoff plate/ damped wave

Ω is an open set of $\mathbb{R}^2$ with regular boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$,
The constant $\gamma > 0$ is the rotational inertia of the plate and the constant $0 < \mu < \frac{1}{2}$ is the Poisson coefficient.
The boundary operators B_1, B_2 are defined by

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$

$$B_2 u = \partial_\tau \left((\nu_1^2 - \nu_2^2) u_{xy} + \nu_1\nu_2 (u_{yy} - u_{xx}) \right),$$

where $\nu = (\nu_1, \nu_2)$ is the unit outer normal vector to Γ and $\tau = (-\nu_2, \nu_1)$ is a unit tangent vector.

Energy estimates.

Introduce the energy, (setting $P_\gamma u = u - \gamma \Delta u$)

$$E(t) = \frac{1}{2} \int_\Omega \{ |P_\gamma^{\frac{1}{2}} u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv \}(x,t)\, dx.$$

We have:

$$E(t) \leq \frac{C_\alpha}{(t+1)^{\frac{1}{3}}} \left(||u^0||^2_{H^3(\Omega)} + ||u^1||^2_{H^2(\Omega)} + ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H^1_0(\Omega)} \right).$$

FDM, interpolation, good choice of functional inequalities,
Borichev-Tomilov criterion.

Damped Kirchhoff plate/ undamped wave

$$
\begin{cases}
u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + \alpha v + u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\
v_{tt} - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times (0, \infty) \\
u = \partial_\nu u = 0 & \text{on} \quad \Gamma_0 \times (0, \infty) \\
\Delta u + (1 - \mu) B_1 u = 0 & \text{on} \quad \Gamma_1 \times (0, \infty) \\
\partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1 - \mu) B_2 u = 0 & \text{on} \quad \Gamma_1 \times (0, \infty) \\
v = 0 & \text{on} \quad \Gamma \times (0, \infty) \\
u(0) = u^0 \in V, \quad u_t(0) = u^1 \in H_0^1(\Omega), & \\
v(0) = v^0 \in H_0^1(\Omega), \quad v_t(0) = v^1 \in L^2(\Omega). &
\end{cases}
$$

Energy estimates.

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{|P_\gamma u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv\}(x,t)\,dx.$$

We have:

$$E(t) \leq \frac{C_\alpha}{(t+1)^{\frac{1}{4}}}\left(||u^0||^2_{H^3(\Omega)} + ||u^1||^2_{H^2(\Omega)} + ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H^1_0(\Omega)}\right).$$

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!