Simultaneous and indirect control of waves: some recent developments and open problems

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Open problems

- Simultaneous Control

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  - A system of wave equations

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  - Lamé systems with localized damping

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  - A plate-wave system with localized damping

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- Indirect Control

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  - Mindlin-Timoshenko plate

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  - A coupled system of waves with different principal operators
  - Mindlin-Timoshenko plate
  - Kirchhoff plate-wave

A system of uncoupled wave equation

Consider the system of uncoupled wave equations

$$
\begin{aligned}
&u_{jtt} - a_j \Delta u_j = 0 \text{ in } Q \\
&u_j = 0 \text{ on } \Gamma \times (0, T) \\
&u_j(x,0) = u_j^0(x), \quad u_{jt}(x,0) = u_j^1(x) \text{ in } \Omega, \quad j = 1,\ 2, ...,\ q,
\end{aligned}
$$

where $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$ for each j.

1988: Haraux (1988) shows for arbitrary nonempty open set ω:

If $\sum_{j=1}^{q} u_j(x,t) = 0$ in $\omega \times (0, T)$ then $u_j^0 = 0, \quad u_j^1 = 0$ in $\Omega, \quad \forall j$.
provided that $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.
Question: Can we quantify that unique continuation result?

For a single wave equation, we have the weak observability inequality [Bellassoued, 2005]:

$$||u^0||^2_{L^2(\Omega)} + ||u^1||^2_{H^{-1}(\Omega)} \leq \frac{c\Big\{||u^0||^2_{L^2(\Omega)} + ||u^1||^2_{H^{-1}(\Omega)}\Big\}}{\lambda} + Ce^{\mu\lambda}\int_0^T r(t)^2 \int_{\omega_0} |u(x,t)|^2\, dxdt.$$

For the uncoupled system, assuming $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$, do we have:

$$\sum_{j=1}^q \Big\{||u_j^0||^2_{L^2(\Omega)} + ||u_j^1||^2_{H^{-1}(\Omega)}\Big\} \leq \frac{c\sum_{j=1}^q \Big\{||u_j^0||^2_{H_0^1(\Omega)} + ||u_j^1||^2_{L^2(\Omega)}\Big\}}{\lambda} + Ce^{\mu\lambda}\int_0^T r(t)^2 \int_{\omega_0} |\sum_{j=1}^q u_j(x,t)|^2\, dxdt?$$

Carleman estimate

Introduce a function $q \in C^2(\bar{\Omega})$ satisfying:

i) ii) $\min\left\{|\nabla q(x)|; x \in \bar{\Omega}\right\} > 0.$

iii) $\frac{1}{4}\min\{a_j; j = 1,\ 2, ...,\ q\}|\nabla q(x)|^2 \geq R_1^2 \geq R_0^2 > 0, \quad \forall x \in \bar{\Omega},$

where $R_0 = \min\left\{\sqrt{q(x)}; x \in \bar{\Omega}\right\}$, and $R_1 = \max\left\{\sqrt{q(x)}; x \in \bar{\Omega}\right\}$. Let ν be the unit normal pointing into the exterior of Ω, and set

$$\Gamma_0 = \left\{x \in \partial\Omega; \nu_j q_{x_j}(x) > 0\right\}.$$

Let $T > 2R_1$. Choose a constant $\mu \in (0, 1)$ such that

$$(2R_1/T)^2 < \mu < 2R_1/T.$$

Set $\varphi(x,t) = d(x) - \mu(t - T/2)^2$. Set
Define a differential operator $\mathcal{P}$ by $\mathcal{P}u = u_{tt} - (b_{ij}(x)u_{x_j})_{x_i}$.
For a single wave equation, we have the Carleman estimates:

Lemma 1. (Carleman estimates)

Assume that Γ_0 is defined as above and ω is a neighborhood of Γ_0.
i) [Fu-Yong-Zhang] Then there exists $\lambda_0 > 1$ and a positive constant $C = C(\Omega, T)$, such that for all $\lambda \geq \lambda_0$ and any $u \in H_0^1(Q)$ with $\mathcal{P}u \in L^2(Q)$, it holds:

$$\lambda \int_Q e^{2\lambda\varphi}(\lambda^2|u|^2 + |u_t|^2 + |\nabla u|^2)\, dxdt \leq C||e^{\lambda\varphi}(\mathcal{P}u)||^2_{L^2(Q)} + C\lambda^2 \int_0^T \int_\omega e^{2\lambda\varphi}(\lambda^2|u|^2 + |u_t|^2)\, dxdt.$$

ii) [Duyckaerts-Zhang-Zuazua] Let $V \in L^{\infty}(0, T; L^{m}(\Omega))$ with $m \in [N, +\infty)$. Then there exist $\lambda_0 > 1$ and a positive constant $C = C(\Omega, T)$, such that for all $\lambda \geq \lambda_0$ and any $u \in C([0, T]; L^2(\Omega))$ satisfying $u(x, 0) = u(x, T) = 0$ for $x \in$, $\mathcal{P}u \in H^{-1}(Q)$, and

$$(u, \mathcal{P}\eta) = \langle \mathcal{P}u, \eta \rangle_{H^{-1}(Q), H_0^1(Q)}, \quad \forall \eta \in H_0^1(Q) \text{ with } \mathcal{P}\eta \in L^2(Q),$$

it holds:

$$\lambda ||e^{\lambda\varphi} u||^2_{L^2(Q)} \leq C \Big(||e^{\lambda\varphi}(\mathcal{P}u - Vu)||^2_{H^{-1}(Q)}$$
$$+ \frac{1}{\lambda^{(2-2N/m)}} ||e^{\lambda\varphi} Vu||^2_{L^2(0,T;H^{-N/m}(\Omega))} + \lambda^2 ||e^{\lambda\varphi} u||^2_{L^2(0,T;L^2(\omega))} \Big).$$

For the uncoupled system, do we have, assuming $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$:
There exist $\lambda_0 > 1$ and a positive constant $C = C(\Omega, T)$, such that for all $\lambda \geq \lambda_0$ and any $u \in \left[H_0^1(Q)\right]^q$ with $\mathcal{P}u \in \left[L^2(Q)\right]^q$, it holds:

$$\lambda \sum_{j=1}^{q} \int_Q e^{2\lambda\varphi}(\lambda^2|u_j|^2 + |u_{jt}|^2 + |\nabla u_j|^2)\, dxdt \leq C \sum_{j=1}^{q} ||e^{\lambda\varphi}(\mathcal{P}u_j)||^2_{L^2(Q)}$$
$$+C\lambda^2 \int_0^T \int_\omega e^{2\lambda\varphi} \left(\lambda^2|\sum_{j=1}^{q} u_j|^2 + |\sum_{j=1}^{q} u_{jt}|^2\right) dxdt?$$

Wave equations: simultaneous boundary damping

Consider the system of damped wave equations (to fix ideas, we assume the damping on the whole boundary):

$$
\begin{aligned}
& w_{jtt} - a_j \Delta w_j = 0 \text{ in } Q \\
& a_j \partial_\nu w_j + c w_j + d \sum_{j=1}^{q} w_{jt} = 0 \text{ on } \Gamma \times (0, T) \\
& w_j(x,0) = w_j^0(x), \quad w_{jt}(x,0) = w_j^1(x) \text{ in } \Omega, \quad j = 1,\ 2, ...,\ q,
\end{aligned}
$$

where $(w_j^0, w_j^1) \in H^1(\Omega) \times L^2(\Omega)$

Assuming that c and d are positive constants, what kind of stability should we expect for the underlying semigroup?

Lamé systems with boundary damping

Given $(y_j^0, y_j^1)_j \in \left(\left[H_0^1(\Omega)\right]^N \times \left[L^2(\Omega)\right]^N\right)^q$, and a function $d \in L^\infty(\Omega)$, $d \geq 0$, consider the damped elastodynamic system

$$
\begin{aligned}
& y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j)\nabla \mathrm{div}(y_j) = 0 \text{ in } \Omega \times (0,\infty) \\
& \mu_j \partial_\nu y_j + (\mu_j + \lambda_j)\mathrm{div}(y_j)\nu + c y_j + d \sum_{k=1}^{q} y_{kt} = 0 \text{ on } \Gamma \times (0,\infty) \\
& y_j(x,0) = y_j^0(x), \quad y_{jt}(x,0) = y_j^1(x), \text{ in } \Omega, \\
& j = 1,\ 2, ...,\ q,
\end{aligned}
$$

where, for each j, μ_j and λ_j are the Lamé constants, and c and d are positive constants.

Lamé systems with localized damping

The total energy is given, for all $t \geq 0$, by

$$2E(t) = \sum_{j=1}^{q} \int_{\Omega} \{|y_{jt}(x,t)|^2 + \mu_j |\nabla y_j(x,t)|^2 + (\mu_j + \lambda_j)|\mathrm{div}(y_j(x,t))|^2\}\, dx$$
$$+c \sum_{j=1}^{q} \int_{\Gamma} |y_j(\gamma,t)|^2\, d\gamma.$$

E is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_{\Gamma} d \left| \sum_{k=1}^{q} y_{kt}(\gamma,t) \right|^2 d\gamma.$$

Lamé systems with localized damping

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$$+c \sum_{j=1}^{q} \int_{\Gamma} |y_j(\gamma,t)|^2 \, d\gamma.$$

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Question 1: Does the energy E decay to zero as time goes to infinity?

Lamé systems with localized damping

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$$+c \sum_{j=1}^{q} \int_{\Gamma} |y_j(\gamma,t)|^2\, d\gamma.$$

E is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_{\Gamma} d \left| \sum_{k=1}^{q} y_{kt}(\gamma,t) \right|^2 d\gamma.$$

Question 1: Does the energy E decay to zero as time goes to infinity?
Question 2: Under which conditions is the Lamé system exponentially stable?

A Plate-wave system

Consider the damped system

$$
\begin{cases}
y_{tt} - \Delta y + d(x)(y_t + z_t) = 0 \text{ in } \Omega \times (0, \infty) \\
z_{tt} - \gamma \Delta z_{tt} + \Delta^2 z + d(x)(y_t + z_t) = 0 \text{ in } \Omega \times (0, \infty) \\
y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0, \infty) \\
y(0) = y^0 \in H_0^1(\Omega), \quad y_t(0) = y^1 \in L^2(\Omega), \\
z(0) = z^0 \in H_0^2(\Omega), \quad z_t(0) = z^1 \in L^2(\Omega).
\end{cases}
$$

The total energy is given, for all $t \geq 0$, by

$$
2E(t) = \int_\Omega \{ |y_t(x,t)|^2 + |\nabla y(x,t)|^2 + |z_t(x,t)|^2 + |\Delta z(x,t)|^2 \} \, dx,
$$

A Plate-wave system

E is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_{\Omega} d(x)\,|y_t(x,t) + z_t(x,t)|^2 \; dx.$$

A Plate-wave system

E is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_\Omega d(x) \left| y_t(x,t) + z_t(x,t) \right|^2 \, dx.$$

For this coupled system we have the following strong stability and exponential stability result:

Euler-Bernoulli Plate-wave system: $\gamma = 0$

Theorem 1: EECT 2013. Strong stability

Let ω be an arbitrary nonvoid open set contained in Ω. Suppose that the damping coefficient d is positive in ω.
The system is strongly stable:

$$\lim_{t \to \infty} E(t) = 0$$

provided that either meas$(\partial\omega \cap \partial\Omega) > 0$, or else, the only solution of $\Delta u = -u$ in Ω and $u = 0$ on $\partial\Omega$ is $u = 0$.

Euler-Bernoulli Plate-wave system

Theorem 2: EECT 2013 Exponential stability

Assume that Let $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(z^0, z^1) \in H_0^2(\Omega) \times L^2(\Omega)$.
Assume that ω satisfies the Liu geometric control condition (SICON 1997), and suppose that the damping is effective in ω:

$$\exists d_0 > 0 : d(x) \geq d_0 \text{ a.e. } \omega.$$

There exist positive constants M and κ, independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t} E(0), \text{ for all } t \geq 0.$$

An observability inequality

Let $T > 0$. Let ω be a nonempty open set in Ω satisfying the Liu geometric control condition.
Consider the uncoupled system

$$\begin{cases} y_{tt} - \Delta y = 0 \text{ in } \Omega \times (0, T) \\ z_{tt} + \Delta^2 z = 0 \text{ in } \Omega \times (0, T) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0, T). \end{cases}$$

There exists $T_0 > 0$ such that for any $T > T_0$, there exists $C > 0$:

$$E(0) \leq \int_0^T \int_\omega |y_t(x,t) + z_t(x,t)|^2 \, dxdt,$$

Kirchhoff Plate-wave system: $\gamma > 0$

When the plate is the Kirchhoff model, then the exponential decay of the energy is not possible, because the damping coming from the plate is compact relative to the energy space. So, since exponential stability is out of question, what kind of stability do we have?

Plate-wave system with structural/Kelvin-Voigt damping

Consider the new damped system

$$\begin{cases} y_{tt} - \Delta y - \mathrm{div}(d(x)\nabla(y_t + z_t)) = 0 \text{ in } \Omega \times (0,\infty) \\ z_{tt} - \gamma \Delta z_{tt} + \Delta^2 z - \mathrm{div}(d(x)\nabla(y_t + z_t)) = 0 \text{ in } \Omega \times (0,\infty) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0,\infty) \\ y(0) = y^0 \in H_0^1(\Omega), \quad y_t(0) = y^1 \in L^2(\Omega), \\ z(0) = z^0 \in H_0^2(\Omega), \quad z_t(0) = z^1 \in L^2(\Omega). \end{cases}$$

It can be checked that the energy of that system is nonincreasing. Does it decay to zero? If yes, at what rate?

Timoshenko beam

Let $L > 0$, and set $\Omega = (0, L)$, and $\omega = (l_1, l_2)$ with $0 \leq l_1 < l_2 \leq L$. Consider the damped Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty), \end{cases}$$

with the boundary conditions:

(**DD**) $\quad y(0, t) = 0, \quad y(L, t) = 0, \quad z(0, t) = 0, \quad z(L, t) = 0,$ or else

(**DN**) $\quad y(0, t) = 0, \quad y(L, t) = 0, \quad z_x(0, t) = 0, \quad z_x(L, t) = 0, \quad t > 0$

and the initial conditions:

$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$

Timoshenko beam

Let $L > 0$, and set $\Omega = (0, L)$, and $\omega = (l_1, l_2)$ with $0 \leq l_1 < l_2 \leq L$. Consider the damped Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty), \end{cases}$$

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(**DN**) $y(0, t) = 0, \quad y(L, t) = 0, \quad z_x(0, t) = 0, \quad z_x(L, t) = 0, \quad t > 0$

and the initial conditions:

$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$

The damping coefficient a is a nonnegative bounded measurable function, which is positive in ω only.

The energy and main questions

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{\rho_1|y_t(x,t)|^2 + k|y_x(x,t) + z(x,t)|^2\}\,dx$$
$$+\tfrac{1}{2}\int_\Omega \{\rho_2|z_t(x,t)|^2 + \sigma|z_x(x,t)|^2\}\,dx, \quad \forall t \geq 0.$$

The energy E is a nonincreasing function of the time variable t as we have for every $t \geq 0$, (hereafter, $'$ denotes differentiation with respect to time)

$$E'(t) = -\int_\Omega a(x)|y_t(x,t) + z_t(x,t)|^2\,dx.$$

The energy and main questions

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{\rho_1|y_t(x,t)|^2 + k|y_x(x,t) + z(x,t)|^2\}\,dx$$
$$+\frac{1}{2}\int_\Omega \{\rho_2|z_t(x,t)|^2 + \sigma|z_x(x,t)|^2\}\,dx, \quad \forall t \geq 0.$$

The energy E is a nonincreasing function of the time variable t as we have for every $t \geq 0$, (hereafter, $'$ denotes differentiation with respect to time)

$$E'(t) = -\int_\Omega a(x)|y_t(x,t) + z_t(x,t)|^2\,dx.$$

As before, our main purpose is to answer the following questions:

- Does the energy $E(t)$ decay to zero as the time variable t goes to infinity?
- If so, how fast? And if not, why?

Timoshenko beam

Theorem 3: Strong stability (CRAS 2015)

Suppose that ω is an arbitrary nonempty open interval in Ω. Let the damping coefficient a be positive in ω. In either of the (**DD**) or (**DN**) case, the associated system is strongly stable:

$$\lim_{t \to \infty} E(t) = 0$$

if and only if $\partial\omega \cap \partial\Omega \neq \emptyset$.

Timoshenko beam

Theorem 4: Exponential stability (CRAS 2015)

Suppose that ω is an arbitrary nonempty open interval in Ω with $\partial\omega \cap \partial\Omega \neq \emptyset$. Let the damping coefficient a satisfy

$$a(x) \geq a_0 > 0, \text{ a.e. in } \omega.$$

There exist positive constants M and κ, independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t}E(0), \text{ for all } t \geq 0.$$

Timoshenko beam with structural damping

Let $L > 0$, and set $\Omega = (0, L)$, and $\omega = (l_1, l_2)$ with $0 \leq l_1 < l_2 \leq L$. Consider the damped Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x - (a(x)(y_{xt} + z_t))_x = 0 \text{ in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_{xt} + z_t) = 0 \text{ in } (0, L) \times (0, \infty), \end{cases}$$

with the boundary conditions:

(**DD**) $\quad y(0, t) = 0, \quad y(L, t) = 0, \quad z(0, t) = 0, \quad z(L, t) = 0,\ t > 0$

and the initial conditions:

$$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$$

Timoshenko beam with structural damping

Let $L > 0$, and set $\Omega = (0, L)$, and $\omega = (l_1, l_2)$ with $0 \leq l_1 < l_2 \leq L$. Consider the damped Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x - (a(x)(y_{xt} + z_t))_x = 0 \text{ in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_{xt} + z_t) = 0 \text{ in } (0, L) \times (0, \infty), \end{cases}$$

with the boundary conditions:

(**DD**) $\quad y(0, t) = 0, \quad y(L, t) = 0, \quad z(0, t) = 0, \quad z(L, t) = 0, \ t > 0$

and the initial conditions:

$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$

The damping coefficient a is a nonnegative bounded measurable function, which is positive in ω only.

Whether that system is exponentially stable is unknown, though strong stability is much easier to prove than in the preceding system.

Indirect controllability a coupled system of wave equations with different principal operators

Consider the controllability problems: Given (z^0, z^1) and (w^0, w^1), and $\varepsilon > 0$, find a control h such that if (z, w) solves the system

$$\begin{cases} z_{tt} - \partial_i(b_{ij}(x)\partial_j z) + az + cw - \mathrm{div}(k_{11} z) - (l_{11} z)_t \\ -\mathrm{div}(k_{21} w) - (l_{21} w)_t = h 1_\omega \text{ in } Q \\ \\ w_{tt} - \partial_i(a_{ij}(x)\partial_j w) + bz + dw - \mathrm{div}(k_{12} z) - (l_{12} z)_t \\ -\mathrm{div}(k_{22} w) - (l_{22} w)_t = 0 \text{ in } Q \\ \\ z = 0, \quad w = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ \\ z(0) = z^0; \quad z_t(0) = z^1 \quad w(0) = w^0; \quad w_t(0) = w^1 \text{ in } \Omega, \end{cases}$$

then (exact controllability)

$$z(T) = 0, \quad z_t(T) = 0, \quad w(T) = 0, \quad w_t(T) = 0 \text{ in } \Omega,$$

or else (approximate controllability)

$$||z(T)||_1 + ||z_t(T)||_2 \leq \varepsilon, \quad ||w(T)||_1 + ||w_t(T)||_2 \leq \varepsilon.$$

An earlier attempt to solve that controllability problem was proposed in 2012 by Dehman-Léautaud-Lerousseau: they consider two wave equations coupled in cascade internally when the two principal operators are proportional and Ω is a compact C^∞ manifold with no boundary; in particular they show that, if $\omega \cap \mathcal{O}$ satisfies GCC, then:

$$\widehat{E}(u;0) + E_{-2}(v;0) \leq C \int_0^T \int_\omega |u(x,t)|^2 \, dxdt,$$

where $2E_{-2}(v;0) = ||v^0||^2_{H^{-2}(\Omega)} + ||v^1||^2_{H^{-3}(\Omega)}$.

Observability estimates

Consider the coupled (adjoint) system

$$
\begin{cases}
u_{tt} - \partial_i(b_{ij}(x)\partial_j u) + au + bv + k_{11}\cdot\nabla u + l_{11}u_t \\
+k_{12}\cdot\nabla v + l_{12}v_t = 0 \text{ in } Q \\
\\
v_{tt} - \partial_i(a_{ij}(x)\partial_j v) + cu + dv + k_{21}\cdot\nabla u + l_{21}u_t \\
+k_{22}\cdot\nabla v + l_{22}v_t = 0 \text{ in } Q \\
\\
u = 0, \quad v = 0 \text{ on } \Sigma = \partial\Omega\times(0,T) \\
\\
u(0) = u^0; \quad u_t(0) = u^1 \quad v(0) = v^0; \quad v_t(0) = v^1 \text{ in } \Omega.
\end{cases}
$$

The coupled system is well-posed in $H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times H^{-1}(\Omega)$.
What type of observability inequality do we have for that system?

Introduce the energies:

$$E_u(t) = \frac{1}{2}\int_\Omega \{|u_t(x,t)|^2 + (b_{ij}(x)\partial_j u(x,t)\partial_i u(x,t)\}\, dx,$$

$$\widehat{E}_u(t) = \frac{1}{2}\left(||u(.,t)||^2_{L^2(\Omega)} + ||u_t(.,t)||^2_{H^{-1}(\Omega)}\right).$$

For each $t \in [0, T]$, set

$$E(t) = E_u(t) + \widehat{E}_v(t), \quad \widehat{E}(t) = \widehat{E}_u(t) + \widehat{E}_v(t).$$

Is it possible to impose conditions on the coefficients a_{ij} and b_{ij} such that we have the observability estimate

$$E(0) \leq C\int_0^T\int_\omega (r^2|u_t|^2 + |u|^2)\, dxdt$$

Mindlin-Timoshenko plate

Consider the Mindlin-Timoshenko system with boundary damping

$$
\begin{aligned}
&\rho_1 y_{tt} - k\text{div}(\nabla y + z) = 0 \text{ in } \Omega \times (0,\infty)\\
&\rho_2 z_{tt} - \mu\Delta z - (\lambda+\mu)\nabla\text{div} z + k(\nabla y + z) = 0 \text{ in } \Omega \times (0,\infty)\\
&y = 0, \quad \mu\partial_\nu z + (\lambda+\mu)\text{div}(z)\nu + z_t = 0 \text{ on } \partial\Omega \times (0,\infty)\\
&y(.,0) = y^0 \in H_0^1(\Omega), \quad y_t(.,0) = y^1 \in L^2(\Omega),\\
&z(.,0) = z^0 \in [H_0^1(\Omega)]^N, \quad z_t(.,0) = z^1 \in [L^2(\Omega)]^N.
\end{aligned}
$$

The constants ρ_1, ρ_2, k, and μ are physical constants and are all positive. In particular, the constants λ and μ are the Lamé constants with $\lambda + \mu > 0$.

Mindlin-Timoshenko plate

A work of Nicaise et collaborators (2015) shoes that the indirectly damped Timoshenko beam, ($N = 1$), is only polynomially stable under some parameter constraints when the dissipation occurs only at the left endpoint.
Suppose that:

$$(*) \qquad \frac{k}{\rho_1} = \frac{2\mu + \lambda}{\rho_2}.$$

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Questions: Is the Mindlin-Timoshenko system exponentially stable under $(*)$ when the damping acts on the whole boundary?
What happens when $(*)$ fails?

Kirchhoff plate-wave

Joint work with Ahmed Hajej (U. Cergy-Pontoise, France) and Zayd Hajjej (U. Gabes, Tunisia)

Undamped Kirchhoff plate/ damped wave

Consider the following weakly coupled system of Kirchhoff plate and wave equations:

$$
\begin{cases}
u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + \alpha v = 0 & \text{in} \quad \Omega \times (0,\infty) \\
v_{tt} - \Delta v + d(x) v_t + \alpha u = 0 & \text{in} \quad \Omega \times (0,\infty) \\
u = \partial_\nu u = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\
\Delta u + (1-\mu) B_1 u = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\
\partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1-\mu) B_2 u = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\
v = 0 & \text{on} \quad \Gamma \times (0,\infty) \\
u(0) = u^0 \in V, \;\; u_t(0) = u^1 \in H_0^1(\Omega), & \\
v(0) = v^0 \in H_0^1(\Omega), \;\; v_t(0) = v^1 \in L^2(\Omega). &
\end{cases}
$$

Undamped Kirchhoff plate/ damped wave

Ω is an open set of $\mathbb{R}^2$ with regular boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$,
The constant $\gamma > 0$ is the rotational inertia of the plate and the constant $0 < \mu < \frac{1}{2}$ is the Poisson coefficient.
The boundary operators B_1, B_2 are defined by

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$

$$B_2 u = \partial_\tau \left((\nu_1^2 - \nu_2^2) u_{xy} + \nu_1\nu_2 (u_{yy} - u_{xx}) \right),$$

where $\nu = (\nu_1, \nu_2)$ is the unit outer normal vector to Γ and $\tau = (-\nu_2, \nu_1)$ is a unit tangent vector.

Energy estimates.

Introduce the energy, (setting $P_\gamma u = u - \gamma \Delta u$)

$$E(t) = \frac{1}{2} \int_\Omega \{ |P_\gamma^{\frac{1}{2}} u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv \}(x,t)\, dx.$$

We have:

$$E(t) \leq \frac{C_\alpha}{(t+1)^{\frac{1}{3}}} \left(||u^0||^2_{H^3(\Omega)} + ||u^1||^2_{H^2(\Omega)} + ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H^1_0(\Omega)} \right).$$

Is the energy decay rate kept when the dissipation is now localized on a convenient portion of the domain under consideration? If not, what is the new decay rate?

Damped Kirchhoff plate/ undamped wave

$$
\begin{cases}
u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + \alpha v + d(x)u_t = 0 & \text{in} \quad \Omega \times (0,\infty) \\
v_{tt} - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times (0,\infty) \\
u = \partial_\nu u = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\
\Delta u + (1-\mu)B_1 u = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\
\partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1-\mu)B_2 u = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\
v = 0 & \text{on} \quad \Gamma \times (0,\infty) \\
u(0) = u^0 \in V, \;\; u_t(0) = u^1 \in H_0^1(\Omega), & \\
v(0) = v^0 \in H_0^1(\Omega), \quad v_t(0) = v^1 \in L^2(\Omega). &
\end{cases}
$$

Energy estimates.

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{|p_\gamma u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv\}(x,t)\,dx.$$

We have:

$$E(t) \leq \frac{C_\alpha}{(t+1)^{\frac{1}{4}}}\left(||u^0||^2_{H^3(\Omega)} + ||u^1||^2_{H^2(\Omega)} + ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H^1_0(\Omega)}\right).$$

Is the energy decay rate kept when the dissipation is now localized on a convenient portion of the domain under consideration? If not, what is the new decay rate?

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!