Global Persistence of Geometric Structures for Stratified Euler equations with fractional dissipation

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Stratified Euler equations

The incompressible stratified Euler system is given by the following couples equations :

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla\pi = \theta\vec{e}_2 \\ \partial_t\theta + v\cdot\nabla\theta + (-\Delta)^{\frac{\alpha}{2}}\theta = 0 \\ \mathrm{div} v = 0, \\ (v,\theta)_{|t=0} = (v_0,\theta_0). \end{cases} \qquad (\mathrm{B}_\alpha)$$

Here :

- $v(t,x)\in\mathbb{R}^2 \rightsquigarrow$ is the velocity vector field localized in $x\in\mathbb{R}^2$ at a time $t\geq 0$,
- $\theta(t,x)\in\mathbb{R} \rightsquigarrow$ is the density,
- $\pi(t,x)\in\mathbb{R} \rightsquigarrow$ is the pressure of the fluid,
- $\theta\vec{e}_2 \rightsquigarrow$ is the buoyancy force in the direction $\vec{e}_2 = (0,1)$,
- $(-\Delta)^{\frac{\alpha}{2}} \rightsquigarrow$ is the fractional dissipation,

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\langle x,\xi\rangle}|\xi|^\alpha \widehat{f}(\xi)d\xi, \quad \alpha\in(0,2].$$

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_\alpha \mathrm{P.V}\int_{\mathbb{R}^2}\frac{(f(x)-f(y))}{|x-y|^{2+\alpha}}dy, \quad \alpha\in(0,2),$$

where P.V. designate the principle value and $C_\alpha > 0$.

Formulation vorticity-density

- Vorticity : For any dimension N, the vorticity is defined by

$$\Omega = (\nabla v) - (\nabla v)^t.$$

- Particular case : For $N = 2$ we have $\Omega \rightsquigarrow \omega = \partial_1 v^2 - \partial_2 v^1$.
- Biot-Savart law : $v = \mathcal{N}_2 \star \omega$, where

$$\mathcal{N}_2(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \quad x^\perp = (-x_2, x_1).$$

- Formulation vorticity-density : Applying the curl operator to the 1^{rt} equation of (B_α)

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega = \partial_1\theta, \\ \partial_t\theta + v\cdot\nabla\theta + (-\Delta)^{\frac{\alpha}{2}}\theta = 0, \\ (\omega,\theta)_{|t=0} = (v_0, \theta_0). \end{cases} \qquad (\mathrm{VD}_\alpha)$$

- $\theta = 0$: The system (VD) is reduced to the Euler equations.

An outline on the Euler's equations

- The Euler's equations is given by

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla \pi = 0 \\ \mathrm{div} v = 0, \\ v_{|t=0} = v_0 \end{cases} \tag{E}$$

- Kato : Let $v_0 \in H^s(\mathbb{R}^N)$, with $s > \frac{N}{2} + 1$. Then the system (E) is a locally well-posed, i.e., $v \in C([0, T^\star[, H^s)$ and

$$T^\star < +\infty \Rightarrow \int_0^{T^\star} \|\nabla v(\tau)\|_{L^\infty} d\tau = +\infty. \tag{C-1}$$

- Beale-Kato-Majda : If $T^\star$ is a time of maximal existence, then

$$T^\star < +\infty \Rightarrow \int_0^{T^\star} \|\omega(\tau)\|_{L^\infty} d\tau = +\infty, \quad s > \frac{N}{2} + 1. \tag{C-2}$$

- Remark : (C-1)$\Leftrightarrow$(C-2).
- Particular case : For $N = 2 \Rightarrow \partial_t \omega + v \cdot \nabla \omega = 0$.
  - $\partial_t \omega + v \cdot \nabla \omega = 0 \Leftrightarrow \omega(t,x) = \omega_0(\Psi^{-1}(t,x)) \Rightarrow$ for $t \geq 0$

$$\|\omega(t)\|_{L^p(\mathbb{R}^2)} = \|\omega_0\|_{L^p(\mathbb{R}^2)}, \quad p \in [1, \infty].$$

  - (C-2)$\Rightarrow$ the Kato's sol. is globally well-posed, i.e., $T^\star = +\infty$.

- Yudovich's solutions : If $\omega_0 \in L^2 \cap L^\infty$, then (E) admits a U.G.Sol. $\omega \in L^\infty(\mathbb{R}_+, L^2 \cap L^\infty) + v$ is Log−Lipschitz, $\Psi_t \in C(\mathbb{R}_+ \times \mathbb{R}^2)$ and

$$\Psi_t - \mathrm{Id} \in C^{\exp -Ct}, \quad \text{for } t \geq 0.$$

- Vortex patches : Assume that $\omega_0 \equiv \mathbf{1}_{\Omega_0}$, with Ω_0 is a bounded domain.
  - Transported patch : For $t \geq 0 : \omega(t, \cdot) = \mathbf{1}_{\Omega_t}(\cdot),\ \Omega_t = \Psi(t, \Omega)$.
  - Regularity of Ω_t :
    Question. If $\partial\Omega_0 \in C^{1+\varepsilon}$, with $0 < \varepsilon < 1$. Does $\partial\Omega_t \in C^{1+\varepsilon}$? and what happens for ∇v in L^∞ ?.
- 1rt Think : If we think to apply the Yudovich's theory, we can not conclude any think about the boundary ∂, because the regularity of the flow Ψ_t is degenerating in time.

V. Yudovich : *Non-stationary flows of an ideal incompressible fluid*. Akademija Nauk SSSR. Zhurnal Vycislitelnoi Matematiki i Matematiceskoi Fiziki 3, 1032–1066 (1963).

Chemin's result

Th. 1 : Chemin-1993 Let v_0 be a vector field such that $\mathrm{div} v_0 = 0$ and its vorticity $\omega_0 = \mathbf{1}_{\Omega_0}$ which $\partial\Omega_0 \in C^{1+\varepsilon}$, with $0 < \varepsilon < 1$. Then the system (E) has a U. G. Sol.

- v is lipschtzian function globally in time and $\|\nabla v(t)\|_{L^\infty} \le Ce^{Ct}$,
- $\partial\Omega_t \in C^{1+\varepsilon}$ for every $t \ge 0$.

■ Basic tools :

- Logarithmic estimate :

$$\|\nabla v\|_{L^\infty} \le C\|\omega\|_{L^2\cap L^\infty} \log\left(e + \frac{\|\omega\|_{C^\varepsilon(X)}}{\|\omega\|_{L^\infty}}\right). \qquad \text{(LE)}$$

- Construct an initial family $X_0 = (X_{0,i})_{i\in\{0,1\}}$ from $\partial\Omega_0 \in C^{1+\varepsilon}$ and define the **Push-forward** of X_0 by setting $X_t = (X_{t,i})_{i\in\{0,1\}}$

$$X_{t,i}(x) = (\partial_{X_{0,i}}\Psi_t(x))(\Psi_t^{-1}(x)), \quad \text{for} t \ge 0, i \in \{1,2\}.$$

■ Characterization of X_t : .

- Inhomogeneous transport equation : $\partial_t X_t + v\cdot\nabla X_t = X_t\cdot\nabla v$
- Commutation with $\partial + v\cdot\nabla$: $\partial_t X_t\omega + v\cdot X_t\omega = 0$
- Hölder persistence regularity : $\|X_t\omega(t)\|_{C^\varepsilon} \le C\|X_0\omega_0\|_{C^\varepsilon} e^{CV(t)}$, with $V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$ + (LE) and Gronwall Ineq.

$$\|\nabla v(t)\|_{L^\infty} \le Ce^{Ct}.$$

- Regularity of $\Omega_t = \Psi(t, \Omega_0)$:
  - $\partial\Omega_0 \in C^{1+\varepsilon}$, $\exists(V_0, f_0)$, with $f_0 \in C^{1+\varepsilon}(\mathbb{R}^2), \nabla f_0(x) \neq 0$ on V_0 and $\partial\Omega_0 = f_0^{-1}(\{0\}) \cap V_0$.
  - Let $\chi \in \mathcal{D}(\mathbb{R}^2)$: supp $\chi \subset V_0$ and $\chi \equiv 1$ in $W_0 \supset \partial\Omega_0$.
  - Define

$$X_{0,0} = \nabla^{\perp} f_0, \quad X_{0,1} = (1-\chi)\vec{e}_1.$$

  - We parametrize $\partial\Omega_0$ by the periodic curve $\gamma^0 \in C^{1+\varepsilon}([0, 2\pi]; \mathbb{R}^2)$ as Sol. of ODE

$$\begin{cases} \partial_\sigma \gamma^0(\sigma) = X_{0,0}(\gamma^0(\sigma)) \\ \gamma^0(0) = x_0, \quad x_0 \in \partial\Omega_0. \end{cases}$$

- Evolution parametrization of $\partial\Omega_t$: $\gamma_t(\sigma) = \Psi(t, \gamma_0(\sigma))$.
  - $\partial_\sigma \gamma(t, \sigma) = (X_{0,0}\Psi)(t, \gamma^0(\sigma))$.
  - We have

$$\begin{aligned} \|X_{t,0} \circ \Psi(t)\|_{C^\varepsilon} &\leq \|X_{t,0}\|_{C^\varepsilon} \|\nabla\Psi(t)\|_{L^\infty}^{\varepsilon} \\ &\leq \|X_{t,0}\|_{C^\varepsilon} e^{CV(t)} \\ &\leq C_0, \end{aligned}$$

  - $\partial_{X_{0,0}}\Psi(t) \equiv X_{t,0} \circ \Psi(t) \in L^\infty_{loc}(\mathbb{R}_+, C^\varepsilon)$.
  - $\gamma_t \in L^\infty_{loc}(\mathbb{R}_+, C^\varepsilon)$, so $\partial\Omega_t \in C^{1+\varepsilon}$

1^{er} main result : $\alpha = 2$

■ The vorticity-density formulation for $\alpha = 2$ is

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega = \partial_1\theta \\ \partial_t\theta + v\cdot\nabla\theta - \Delta\theta = 0 \\ (\omega,\theta)_{|t=0} = (\omega_0,\theta_0). \end{cases} \tag{B-2}$$

Th. 2 : Hmidi-Z. If $\partial\Omega_0 \in C^{1+\varepsilon}$, with $0<\varepsilon<1, \omega_0 = \mathbf{1}_{\Omega_0}$ and $\theta\in L^1\cap L^\infty$, with $p > \max\left(\frac{2}{1-\varepsilon}\right)$. Then the (B-2) admits U-G-Sol.

$$(v,\theta)\in L^\infty_{loc}(\mathbb{R}_+;\mathrm{Lip})\times L^\infty(\mathbb{R}_+;L^1\cap L^\infty).$$

More precisely :

(1.1) $\|\nabla v(t)\|_{L^\infty} \le C_0 e^{C_0 t\log^2(2+t)}$,

(1.2) $\omega(t,x) = \mathbf{1}_{\Omega_t}(x) + \widetilde{\theta}(t,x)$, $\widetilde{\theta}\in C^\eta$ for all $\eta<1$,

(1.3) $\|\omega(t)-\mathbf{1}_{\Omega_t}\|_{L^p} \le C_0\log^{2-\frac{2}{p}}(2+t)$.

T. Hmidi and M. Zerguine : *Vortex patch problem for stratified Euler equations*. Comm. Math. Sci. Vol. 12, No. 8, 1541-1563 (2014).

Some remarks are in order.

Rqs. 3

(R.1) Compared to the Chemin's result (Euler equations) our result is not optimal, because

$$\|\omega(t)\|_{L^\infty} \leq C_0 \log^2(2+t). \tag{1}$$

- $\partial_t \omega + v \cdot \omega = \partial_1 \theta \Rightarrow$

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^t \|\nabla\theta(\tau)\|_{L^p} d\tau, \quad p \in [1, \infty],$$

- $\|\nabla\theta\|_{L^1_t L^p} \leq C_0 \log^{2-\frac{2}{p}}(2+t)$.

(R.2) (1)$\Rightarrow$ the additional logarithmic factor in the growth of the gradient of the velocity.

Sketch of the proof : Th. 2

- Introduce $\Gamma = \omega - \partial_1 \Delta^{-1}\theta$ which satisfies

$$\partial_t \Gamma + v \cdot \nabla \Gamma = -[\partial_1 \Delta^{-1}, v \cdot \nabla]\theta.$$

- For $\omega \in L^1 \cap L^p$ and $\theta \in L^2 \cap L^p$, with $p > \frac{2}{1-\varepsilon}$, $0 < \varepsilon < 1$

$$\|[\partial_1 \Delta^{-1}, v \cdot \nabla]\theta\|_{C^\varepsilon} \leq C(\|v\|_{L^2}\|\theta\|_{L^2} + \|\omega\|_{L^2 \cap L^p}\|\theta\|_{L^p}) \leq C_0.$$

- Asymptotic behavior for $\partial_t \theta + v \cdot \nabla \theta - \Delta \theta = 0$.
  - $(1+t)^\beta \|\theta(t)\|_{L^2}^2 \leq C_0$ for $0 < \beta < 1$.
  - $(1+t)\|\theta(t)\|_{L^2}^2 \leq C_0$ for $\beta = 1$.
  - For $\omega \in L^1 \cap L^p$ and $\theta \in L^1 \cap L^p$, with $p \in [2, \infty]$

$$\|\omega(t)\|_{L^\infty} + \|\nabla \theta(t)\|_{L^1_t L^p} \leq C_0 \log^{2-\frac{2}{p}}(2+t), \quad t \geq 0.$$

- For ∇v in L^∞ and the regularity of Ω_t we use Chemin's formalism.

2^{nd} main result : $\alpha = 1$

■ The vorticity-density formulation for $\alpha = 1$ is reads as follows

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega = \partial_1\theta \\ \partial_t\theta + v\cdot\nabla\theta + (-\Delta)^{\frac{1}{2}}\theta = 0 \\ (\omega,\theta)_{|t=0} = (\omega_0,\theta_0). \end{cases} \tag{B_1}$$

Th. 2 : Zerguine. If $\partial\Omega_0 \in C^{1+\varepsilon}$, with $0<\varepsilon<1$ $\omega_0 = \mathbf{1}_{\Omega_0}$ and $\theta \in L^1\cap L^\infty \cap B^{\varepsilon}_{p,\infty}$, with $p > \max\left(\frac{2}{1-\varepsilon}, \frac{2}{\varepsilon}\right)$. Then (B_1) admits a U-G-Sol.

$$(v,\theta) \in L^\infty_{loc}(\mathbb{R}_+; Lip) \times L^\infty(\mathbb{R}_+; L^1\cap L^\infty).$$

To be precise :
(2.1) $\|\nabla v(t)\|_{L^\infty} \le C_0 e^{C_0 t}$,
(2.2) $\omega(t,x) = \mathbf{1}_{\Omega_t}(x) + \widetilde{\theta}(t,x)$, $\quad \widetilde{\theta} \in C^\eta$ for all $\eta < \varepsilon - \frac{2}{p} < 1$,
(2.3) $\|\omega(t) - \mathbf{1}_{\Omega_t}\|_{L^p} \le C_0 \log^{2-\frac{2}{p}}(2+t)$.

M. Zerguine : *The rgular vortex patch problem for stratified Euler equations with critical fractional dissipation.* J .Evol. Equ, Vol. 15, No. 3, 667–698 (2015).

A few remarks are in order.

Rqs. 4

(R.1) The obtained result is sharper than Chemin's result for Euler equationsin the sense that :

$$\|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t}. \tag{2}$$

Due to the fact

- $\|\omega(t)\|_{L^\infty} \leq C_0$, which derived from

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^t \|\nabla\theta(\tau)\|_{L^p} d\tau, \quad p \in [1, \infty],$$

- $\|\nabla\theta\|_{L^1_t L^p} \leq C_0$.

(R.2) The logarithmic losing in the full Laplacian is a low frequency problem. Formally the sylmbol of the fractional laplacian : For $\xi \in B(0,1)$ and $\alpha \in]0,1[\Rightarrow |\xi|^2 << |\xi|^{2\alpha}$.

Sketch of the proof : Th. 3

- Setting $\mathfrak{R} = \partial_1(\Delta)^{-\frac{1}{2}}$ and introduce $\Gamma = \omega - \mathfrak{R}\theta$

$$\partial_t\Gamma + v\cdot\nabla\Gamma = -[\mathfrak{R}, v\cdot\nabla]\theta.$$

  - For $(p,r)\in[2,\infty[\times[1,\infty]$

$$\|[\mathfrak{R}, v\cdot\nabla]\theta\|_{B^0_{p,r}} \leq C\|\nabla v\|_{L^\infty}(\|\theta\|_{B^0_{\infty,r}} + \|\theta\|_{L^p})$$

- For $(p,r)\in[2,\infty[\times]1,\infty[$ and $\varepsilon>0$

$$\|[\mathfrak{R}, v\cdot\nabla]\theta\|_{B^0_{\infty,r}} \leq C(\|\omega\|_{L^\infty} + \|\omega\|_{L^p})(\|\theta\|_{B^0_{\infty,r}} + \|\theta\|_{L^p})$$

- Asymptotic behavior for $\partial_t\theta + v\cdot\nabla\theta + (-\Delta)^{\frac{1}{2}}\theta = 0$.
  - For $p\in[2,\infty[$ and $\theta_0\in L^1\cap L^p$,

$$\|\theta(t)\|_{L^p} \leq \frac{C_0}{(1+t)^{2-\frac{2}{p}}} \quad t\geq 0.$$

  - For $\theta_0\in L^1\cap L^\infty$ and $\theta(t,\cdot)\in H^s$, $s>1$,

$$\|\theta(t)\|_{L^\infty} \leq \frac{C_0}{(1+t)^2} \quad t\geq 0.$$

- For ∇v in L^∞ and the regularity of Ω_t we use Chemin's formalism.

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Thank you for attention.